Econ 204 – Problem Set 2

Due Tuesday, July 25

1. Show that any set in a metric space (X, d) can be written as the intersection of open sets.
2. Suppose the metric space (X, d) is infinite.
   (a) Show that there exists an open set U such that U and U^c are both infinite.
   (b) Show that there exists an infinite subset $Y \subset X$ such that (Y, d) is a discrete metric space.[1]
3. Construct a real function that is continuous at exactly one point.
4. Take any mapping f from a metric space X into a metric space Y. Prove that f is continuous if and only if $f\left(\overline{A}\right) \subseteq \overline{f(A)}$ for every set A. (Hint: use the closed set characterization of continuity).
5. Prove that a metric space (X, d) is discrete if and only if every function on X into any other metric space (Y, ρ), where Y has at least two distinct elements, is continuous.
6. For any continuous function $f : [0, 1] \to \mathbb{R}$, define the functions $T(f) : [0, 1] \to \mathbb{R}$ and $S(f) : [0, 1] \to \mathbb{R}$ by

$$T(f)(x) = 1 + \int_0^x f(s)ds$$

$$S(f)(x) = \begin{cases} f\left(x + \frac{1}{2}\right) & \text{if } x < \frac{1}{2} \\ f(1) & \text{if } x \geq \frac{1}{2} \end{cases}$$

   Let $W(f) = \alpha T(f) + \beta S(f)$ for some $\alpha, \beta \in \mathbb{R}$. Show that if $|\alpha| + |\beta| < 1$, then there exists a continuous function $f : [0, 1] \to \mathbb{R}$ such that $W(f) = f$.

[1] A metric space (X, d) *discrete* if every subset $A \subset X$ is open. Notice that any set equipped with the discrete metric forms a discrete metric space, but not every discrete metric space necessarily has the discrete metric.