Econ 204 – Problem Set 3

Due Friday July 28, 2017 [1]

1. Let (X, d) be a metric space:

   (a) Let $y \in X$ be given. Define the function $d_y : X \to \mathbb{R}$ by

   $$d_y(x) = d(x, y) \tag{1}$$

   Show that d_y is a continuous function on X for each $y \in X$.

   (b) Let A be a subset of X and $x \in X$. Recall that the distance from the point x to the set A is defined as:

   $$\rho(x, A) = \inf \{d(x, a) : a \in A\} \tag{2}$$

   Show that the closure of set A is the set of all points with zero distance to A, that is:

   $$\bar{A} = \{x \in X : \rho(x, A) = 0\} \tag{3}$$

   (c) Now let $A \subset X$ be a compact subset. Show that $\rho(x, A) = d(x, a)$ for some $a \in A$.

2. Let x and y be moving objects in $\mathbb{R}$. Time is discrete, namely $t \in \mathbb{Z}_+ := \{0\} \cup \mathbb{N}$. In addition, $\beta > 1$ is a fixed parameter. For $a, b \in \mathbb{R}$, let $\rho(a, b) := |a - b| \wedge 1$ (as mentioned in the section, the symbol $\wedge$ is sometimes used to refer to the minimum of two elements). Then for any $x, y \in \mathbb{R}^\omega$ [2], let

   $$d(x, y) = \sum_{t \in \mathbb{Z}_+} \beta^{-t} \rho(x_t, y_t) \tag{4}$$

   denotes the distance between $x = (x_0, x_1, \ldots)$ and $y = (y_0, y_1, \ldots)$, where x_t is the position of x at time t on the real line.

   (a) Show that d is a metric on $\mathbb{R}^\omega$.

   (b) Show that $(\mathbb{R}^\omega, d)$ is a bounded metric space.

   (c) Is $[0, 1]^\omega$ an open or closed subset of $\mathbb{R}^\omega$? (in either case present a proof)

   (d) Is $(\mathbb{R}^\omega, d)$ a complete metric space? (prove if yes, otherwise provide a counterexample)

   (e) Is $[0, 1]^\omega$ a totally bounded subset under d? Is it a compact subset?

3. Let X be a connected space. Let $\mathcal{R}$ be an equivalence relation on X such that for each $x \in X$, there exists an open set $\mathcal{O}_x$ containing x such that $\mathcal{O}_x \subset [x]$. Show that $\mathcal{R}$ has only one (distinct) equivalence class.

[1] Please keep your answers short and concise. The solution to each question could well fit in at most one page.

[2] For the definition of $\mathbb{R}^\omega$ please refer to Q3 of ps.1

4. Define the correspondence $\Gamma : [0,1] \to 2^{[0,1]}$ by:

$$\Gamma(x) = \begin{cases} [0,1] \cap \mathbb{Q} & \text{if } x \in [0,1] \setminus \mathbb{Q} \\ [0,1] \setminus \mathbb{Q} & \text{if } x \in [0,1] \cap \mathbb{Q} \end{cases}. \tag{5}$$

Show that Γ is not continuous, but it is lower-hemicontinuous. Is Γ upper-hemicontinuous at any rational? At any irrational? Does this correspondence have a closed graph?

5. Let $f : \mathbb{R}_+ \to \mathbb{R}$. Recall that the left-hand limit of f at $t \in \mathbb{R}_+ \setminus \{0\}$ (if it exists) is

$$\begin{aligned} f(t^-) &= \lim_{s \to t^-} f(s) \\ &= \lim_{s_n \to t} f(s_n) \quad \text{for any sequence } s_n \to t \text{ such that } s_n < t \ \forall n. \end{aligned} \tag{6}$$

(a) Give an example of a function $f : \mathbb{R}_+ \to \mathbb{R}$ that is not continuous but has a finite left-hand limit at some point $s > 0$ (that is, such that $f(s^-)$ exists and $f(s^-) < \infty$).

(b) Suppose $f(t^-)$ exists and is finite for all $t > 0$. Let $f^- : \mathbb{R}_+ \setminus \{0\} \to \mathbb{R}$ be given by $f^-(t) = f(t^-)$. Show that f^- is left-continuous at each $t > 0$. Namely, show that if $t > 0$ and $\{s_n\}$ is a sequence such that $s_n < t \ \forall n$ and $s_n \to t$, then

$$\lim_{n \to \infty} f^-(s_n) = f^-(t) = f(t^-). \tag{7}$$

Hint: When $f(t^-)$ exists and is finite, it is equivalently characterized as follows:

$$f(t^-) = \sup_{s<t} \inf \left\{ f(v) : s \le v < t \right\} = \inf_{s<t} \sup \left\{ f(v) : s \le v < t \right\} \tag{8}$$