Econ 204 – Problem Set 4

Due Tuesday, August 4

1. Let A be an $n \times n$ matrix.
   (a) Show that if λ is an eigenvalue of A then λ^k is an eigenvalue of A^k for $k \in \mathbb{N}$.
   (b) Show that if λ is an eigenvalue of the invertible matrix A then $1/\lambda$ is an eigenvalue of A^{-1}.
   (c) The *eigenspace* of an eigenvalue λ_i is the kernel of $A - \lambda_i I$. Show that the eigenspace of any matrix A belonging to an eigenvalue λ_i is a vector space.
2. Show that similar matrices have the same trace (hint: first show that for any two $n \times n$ matrices A, B, $\operatorname{tr}(AB) = \operatorname{tr}(BA)$).
3. Give an example of a linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ such that $\ker T = \operatorname{im} T$.
4. Call a function $T : V \to W$ *additive* if $T(x + y) = T(x) + T(y)$ for every $x, y \in V$.
   (a) If V, W are vector spaces defined over the field of rational numbers, prove that every additive function is linear.
   (b) Prove that there is some additive function $T : \mathbb{R} \to \mathbb{R}$ that is not linear.
5. Suppose a linear transformation $T : \mathbb{R}^k \to \mathbb{R}^k$ has the property that $T^n = 0$ for some integer $n > 0$. Show that T is not invertible, but that $T + I$ is invertible (hint: the inverse is a polynomial function of T).