1. Let $A$ be an $n \times n$ matrix.
   
   (a) Show that if $\lambda$ is an eigenvalue of $A$ then $\lambda^k$ is an eigenvalue of $A^k$ for $k \in \mathbb{N}$.
   
   (b) Show that if $\lambda$ is an eigenvalue of the invertible matrix $A$ then $1/\lambda$ is an eigenvalue of $A^{-1}$.
   
   (c) The eigenspace of an eigenvalue $\lambda_i$ is the kernel of $A - \lambda_i I$. Show that the eigenspace of any matrix $A$ belonging to an eigenvalue $\lambda_i$ is a vector space.

2. Show that similar matrices have the same trace (hint: first show that for any two $n \times n$ matrices $A, B$, $\text{tr}(AB) = \text{tr}(BA)$).

3. Give an example of a linear transformation $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ such that $\text{ker} T = \text{im} T$.

4. Call a function $T : V \rightarrow W$ additive if $T(x + y) = T(x) + T(y)$ for every $x, y \in V$.
   
   (a) If $V, W$ are vector spaces defined over the field of rational numbers, prove that every additive function is linear.
   
   (b) Prove that there is some additive function $T : \mathbb{R} \rightarrow \mathbb{R}$ that is not linear.

5. Suppose a linear transformation $T : \mathbb{R}^k \rightarrow \mathbb{R}^k$ has the property that $T^n = 0$ for some integer $n > 0$. Show that $T$ is not invertible, but that $T + I$ is invertible (hint: the inverse is a polynomial function of $T$).