Econ 204 – Problem Set 5

Due Friday August 4, 2017 [1]

1. Assume that $f : [0, \infty) \to \mathbb{R}$ is differentiable for all $x > 0$, and $f'(x) \to 0$ as $x \to \infty$. Prove
$$\lim_{x\to\infty} [f(x+1) - f(x)] \to 0. \tag{1}$$
Hint: Use the mean value theorem, and then send $x \to \infty$.

2. Let $F : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by $F(x, y) = \big(e^y \cos(x), e^y \sin(x)\big)$.
   (a) Show that F satisfies the prerequisites of the Inverse Function Theorem for all $(x, y) \in \mathbb{R}^2$ (and is therefore locally injective everywhere) but F is not globally injective.
   (b) Compute the Jacobian of the local inverse of F and evaluate it at $F\left(\frac{\pi}{3}, 0\right)$.
   (c) Find an explicit formula for the continuous inverse of F mapping a neighborhood of $F\left(\frac{\pi}{3}, 0\right)$ into a neighborhood of $\left(\frac{\pi}{3}, 0\right)$ and verify that its Jacobian at $F(\frac{\pi}{3}, 0)$ equals the one you calculated in (ii).

3. Let $f_n : \mathbb{R} \to \mathbb{R}$ be differentiable for each $n \in \mathbb{N}$ with $\left|f_n'(x)\right| \le 1$ for all n and x. Assume,
$$\lim_{n\to\infty} f_n(x) = g(x) \tag{2}$$
for all x. Prove that $g : \mathbb{R} \to \mathbb{R}$ is Lipschitz-continuous.

4. The goal of this exercise is to verify the **Banach-Steinhaus** theorem. Let $\{T_n\}$ be a sequence of bounded linear functions $T_n : X \to Y$ from a Banach (complete normed vector) space X into a normed vector space Y, such that $\{T_n(x)\}$ is bounded for every $x \in X$, that is for all $x \in X$ there exists $c_x \in \mathbb{R}_+$ such that:
$$\big\|T_n(x)\big\| \le c_x \quad \forall n \in \mathbb{N} \tag{3}$$
Then, we want to show that the sequence of norms $\big\{\|T_n\|\big\}$ is bounded, that is there exists $c > 0$ such that $\|T_n\| \le c$ for all $n \in \mathbb{N}$.
   (a) For every $k \in \mathbb{N}$ let $A_k \subseteq X$ be the set of all $x \in X$ such that $\big\|T_n(x)\big\| \le k$ for all n. Show that A_k is closed under the X-norm.
   (b) Use equation (3) to show that $X = \bigcup_{k\in\mathbb{N}} A_k$.
   (c) The **Baire's** theorem states that in this case since X is complete, there exists some A_{k_0} that contains an open ball, say $B(x_0, \varepsilon) \subseteq A_{k_0}$. Take this result as given, and prove there exists some constant $c > 0$ such that
$$\|T_n\| \le c \quad \forall n \in \mathbb{N}. \tag{4}$$
   Hint: For every nonzero $x \in X$ there exists $\gamma > 0$ such that $x = \frac{1}{\gamma}(z - x_0)$, where $x_0, z \in B(x_0, \varepsilon)$ and $\gamma > 0$.

[1] Please keep your answers short and concise. The solution to each question could well fit in at most one page.