Econ 204 – Problem Set 6
Due Monday, August 4 at noon in Walker’s mailbox

1. Call a vector \( \pi \in \mathbb{R}^n \) a probability vector if
   \[
   \sum_{i=1}^{n} \pi_i = 1 \quad \text{and} \quad \pi_i \geq 0 \quad \forall i
   \]
   We say there are \( n \) states of the world, and \( \pi_i \) is the probability that
   state \( i \) occurs. Suppose there are two traders (trader 1 and trader 2) who
   each have a set of prior probability distributions \( (\Pi_1, \Pi_2) \) which are
   nonempty, convex, and compact. Call a trade a vector \( f \in \mathbb{R}^n \), which
   denotes the net transfer trader 1 receives in each state of the world (and
   thus \( -f \) is the net transfer trader 2 receives in each state of the world).
   A trade is agreeable if
   \[
   \inf_{\pi \in \Pi_1} \sum_{i=1}^{n} \pi_i f_i > 0 \quad \text{and} \quad \inf_{\pi \in \Pi_2} \sum_{i=1}^{n} \pi_i (-f_i) > 0
   \]
   Prove that there exists an agreeable trade if and only if there is no common
   prior (that is, \( \Pi_1 \cap \Pi_2 = \emptyset \)).

2. Let \( A \) be a nonempty, compact and convex subset of \( \mathbb{R}^2 \) such that if
   \((x, y) \in A \) for some \( x, y \in \mathbb{R} \) then there exists some \( z \in \mathbb{R} \) such
   that \((y, z) \in A \). Prove that \((x^*, x^*) \in A \) for some \( x^* \in \mathbb{R} \).

3. Show that the closure of a convex set is convex.

4. Let \( f : \mathbb{R}^2 \to \mathbb{R}^2 \) be a \( C^1 \) function and define \( F : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}^2 \) by
   \[
   F(x, \omega) = f(x_1, x_2) + 2(\omega_1 + \omega_2, \omega_2) + 3(\omega_1, \omega_2^2).
   \]
   Show that there is a set of Lebesgue measure zero, \( \Omega_0 \subset \mathbb{R}^2 \), such that if
   \( \omega \notin \Omega_0 \), then for each \( x_0 \) satisfying \( F(x_0, \omega_0) = 0 \) there is an open set \( U \)
   containing \( x_0 \), an open set \( V \) containing \( \omega_0 \), and a \( C^1 \) function \( h : V \to U \)
   such that for all \( \omega \in V \), \( x = h(\omega) \) is the unique element of \( U \)
   satisfying \( F(x, \omega) = 0 \).

5. Define an open half-space as \( S = \{ y \in \mathbb{R}^n : p \cdot y < c \} \) for some \( p \in \mathbb{R}^n \)
   and \( c \in \mathbb{R} \). Show that if \( B \subset \mathbb{R}^n \) is open and convex, then \( B = \cap_{i \in I} S_i \)
   where \( \{S_i : i \in I\} \) is the set of all such open half-spaces containing \( B \).

6. Consider the following system of first order differential equations:
   \[
   \begin{align*}
   \dot{x} &= x^{1/4} - y \\
   \dot{y} &= y[3/2 x^{-2/3} - 1/x^2]
   \end{align*}
   \]
   (a) Plot the \( \dot{x} = 0 \) and \( \dot{y} = 0 \) loci for \( x > 0 \) in a phase diagram. Show the
   steady state, the direction of motion, and the approximate location
   of the stable and unstable arms.
(b) Linearize the system using a Taylor-series expansion around the $x > 0$ steady state. Write down the linearized equations.

(c) Plot a phase diagram for the linearized system and compare the behavior at the steady state of the two systems.

(d) Give the general solution of the linearized system.