Econ 204 – Problem Set 6

Due Monday, August 4 at noon in Walker's mailbox

1. Call a vector $\pi \in \mathbb{R}^n$ a *probability vector* if

$$\sum_i^n \pi_i = 1 \text{ and } \pi_i \geq 0 \; \forall i$$

   We say there are n states of the world, and π_i is the probability that state i occurs. Suppose there are two traders (trader 1 and trader 2) who each have a set of prior probability distributions (Π_1 and Π_2) which are nonempty, convex, and compact. Call a *trade* a vector $f \in \mathbb{R}^n$, which denotes the net transfer trader 1 receives in each state of the world (and thus $-f$ is the net transfer trader 2 receives in each state of the world). A trade is *agreeable* if

$$\inf_{\pi \in \Pi_1} \sum_{i=1}^n \pi_i f_i > 0 \quad \text{and} \quad \inf_{\pi \in \Pi_2} \sum_{i=1}^n \pi_i (-f_i) > 0$$

   Prove that there exists an agreeable trade if and only if there is no common prior (that is, $\Pi_1 \cap \Pi_2 = \varnothing$).

2. Let A be a nonempty, compact and convex subset of $\mathbb{R}^2$ such that if $(x, y) \in A$ for some $x, y \in \mathbb{R}$ then there exists some $z \in \mathbb{R}$ such that $(y, z) \in A$. Prove that $(x^*, x^*) \in A$ for some $x^* \in \mathbb{R}$.

3. Show that the closure of a convex set is convex.

4. Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be a C^1 function and define $F : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}^2$ by

$$F(x, \omega) = f(x_1, x_2) + 2(\omega_1 + \omega_2, \omega_2) + 3(\omega_1^3, \omega_2^3).$$

   Show that there is a set of Lebesgue measure zero, $\Omega_0 \subset \mathbb{R}^2$, such that if $\omega \notin \Omega_0$, then for each x_0 satisfying $F(x_0, \omega_0) = 0$ there is an open set U containing x_0, an open set V containing ω_0, and a C^1 function $h : V \to U$ such that for all $\omega \in V$, $x = h(\omega)$ is the unique element of U satisfying $F(x, \omega) = 0$.

5. Define an open half-space as $S = \{y \in \mathbb{R}^n : p \cdot y < c\}$ for some $p \in \mathbb{R}^n$ and $c \in R$. Show that if $B \subset \mathbb{R}^n$ is open and convex, then $B = \cap_{i \in I} S_i$ where $\{S_i : i \in I\}$ is the set of all such open half-spaces containing B.

6. Consider the following sytsem of first order differential equations:

$$\begin{aligned} \dot{x} &= x^{1/4} - y \\ \dot{y} &= y[\tfrac{3}{2} x^{-2/3} - \tfrac{1}{10}] \end{aligned}$$

   (a) Plot the $\dot{x} = 0$ and $\dot{y} = 0$ loci for $x > 0$ in a phase diagram. Show the steady state, the direction of motion, and the approximate location of the stable and unstable arms.

(b) Linearize the system using a Taylor-series expansion around the $x > 0$ steady state. Write down the linearized equations.

(c) Plot a phase diagram for the linearized system and compare the behavior at the steady state of the two systems.

(d) Give the general solution of the linearized system.