Economics 204 Summer/Fall 2017
Final Exam

Answer all of the questions below. Be as complete, correct, and concise as possible. There are 6 questions for a total of 165 points possible; point values for each problem are in parentheses. For questions with subparts, each subpart is worth the same number of points. You have 180 minutes to complete the exam. Use the points as a guide to allocating your time. You may use any result from class with appropriate references unless you are specifically being asked to prove it.

Do not turn the page until the exam begins.

1. (15) Define or state each of the following.

   (a) continuous function $f : X \to Y$ from a metric space (X, d) to a metric space (Y, ρ)

   (b) eigenvalue of a linear transformation

   (c) Separating Hyperplane Theorem

2. (30) Prove that for every $n \in \mathbf{N} = \{1, 2, 3, \ldots\}$,

$$\sum_{k=1}^{n} \frac{1}{k(k+1)} = \frac{n}{n+1}$$

3. (30) Let X and Y be vector spaces over the same field F. Let $T : X \to Y$ be a linear transformation such that $\ker T = \{0\}$. Show that if the set V is a basis of X, then the set $T(V) = \{T(v) : v \in V\}$ is a basis of $\operatorname{Im} T$.

4. (30) Let $f : \mathbf{R}^n \times \mathbf{R}^p \to \mathbf{R}^n$ be continuous.

   (a) Let $C \subseteq \mathbf{R}^n$ be compact. Let $\Psi : \mathbf{R}^p \to 2^C$ be given by

   $$\Psi(a) = \{x \in C : f(x, a) = 0\}$$

   Show that Ψ is upper hemicontinuous.

   (b) Let $X \subseteq \mathbf{R}^n$ and $A \subseteq \mathbf{R}^p$ be open sets, and let $f : X \times A \to \mathbf{R}^n$ be C^1. Let $\Psi : A \to 2^X$ be given by

   $$\Psi(a) = \{x \in X : f(x, a) = 0\}$$

   Suppose 0 is a regular value of $f(\cdot, a^*)$, that is, for each x^* such that $f(x^*, a^*) = 0$, $\det(D_x f(x^*, a^*)) \neq 0$. Show that Ψ is lower hemicontinuous at a^*.

5. (30) Let $X \subseteq \mathbf{R}^n$ be open and $f : X \to \mathbf{R}$ be differentiable. Let $C \subseteq X$ be a compact, convex set such that for each $x \in C$, there exists $\varepsilon_x > 0$ and $M_x \geq 0$ such that $\|Df(y)\| \leq M_x$ for all $y \in B_{\varepsilon_x}(x)$. Show that f is Lipschitz continuous on C.

   (**Hint:** Use the open cover definition of compactness.)

6. (30) Let (X, d) be a metric space, and $C \subseteq X$ be a nonempty compact set. Let $f : C \to C$ be a function such that

   $$d(f(x), f(y)) < d(x, y) \ \ \forall x, y \in C \text{ such that } x \neq y$$

   Show that f has a unique fixed point in C.

   (**Hint:** No theorem from class will immediately imply this result.)