Announcements:
- PS 3 due now
  → solutions ≈ 2pm today
- PS 4 posted
  → due Tuesday
- last year's exam posted ≈ Sunday

# Econ 204 2018

## Lecture 10

## Outline

1. Diagonalization of Real Symmetric Matrices
2. Application to Quadratic Forms
3. Linear Maps Between Normed Spaces

# How Might This Matter

$c_{t+1} = b_{11} c_t + b_{12} k_t$
$k_{t+1} = b_{21} c_t + b_{22} k_t$
$\forall t$

- Why does diagonalizability matter?

  Consider a two-dimensional linear difference equation:

$$\begin{pmatrix} c_{t+1} \\ k_{t+1} \end{pmatrix} = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} \begin{pmatrix} c_t \\ k_t \end{pmatrix} \quad \forall t = 0, 1, 2, 3, \ldots$$

  given an initial condition $c_0, k_0$, or, setting

$$y_t = \begin{pmatrix} c_t \\ k_t \end{pmatrix} \quad \forall t \quad \text{and} \quad B = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}$$

  we can rewrite this more compactly as

$$y_{t+1} = By_t \quad \forall t$$

  where $b_{ij} \in \mathbf{R}$ each $i, j$.

We want to find a solution $y_t$, $t = 1, 2, 3, \ldots$ given initial condition $y_0$. (Why?)

Such a dynamical system will arise for example as a characterization of the solution to a standard infinite-horizon optimal growth problem (202a, lecture 2).

If $B$ is diagonalizable, this can be easily solved after a change of basis. If $B$ is diagonalizable, choose an invertible $2 \times 2$ real matrix $P$ such that

$$P^{-1}BP = D = \begin{pmatrix} d_1 & 0 \\ 0 & d_2 \end{pmatrix}$$

Then

$$\begin{aligned} y_{t+1} = By_t \quad \forall t \quad &\iff \quad P^{-1}y_{t+1} = P^{-1}By_t \quad \forall t \qquad \text{(mult. by } P^{-1}\text{)} \\ &\iff \quad \left[P^{-1}y_{t+1}\right] = \left(P^{-1}BP\right)\left[P^{-1}y_t\right] \quad \forall t \qquad PP^{-1} = I \\ &\iff \quad \bar{y}_{t+1} = D\bar{y}_t \quad \forall t \\ &\qquad\qquad = \begin{pmatrix} d_1 & 0 \\ 0 & d_2 \end{pmatrix} \bar{y}_t \end{aligned}$$

where $\bar{y}_t = P^{-1} y_t \quad \forall t$

$\Leftrightarrow \quad \bar{y}_{i\,t+1} = d_i \, \bar{y}_{it} \quad \forall t \;, i=1,2$

where $\bar{y}_t = P^{-1} y_t \; \forall t$.

Since $D$ is diagonal, after a change of basis to $\bar{y}_t$, we need to solve two **independent** linear univariate difference equations, which is easy:

$$\bar{y}_{it} = d_i^t \bar{y}_{i0} \quad \forall t$$

- Not all real $n \times n$ matrices are diagonalizable (not even all invertible $n \times n$ matrices are)...so can we identify some classes that are? yesterday:
  - basis of eigenvectors ($\Leftrightarrow$)
  - $n$ distinct eigenvalues ($\Rightarrow$)

- Some types of matrices appear more frequently than others – especially real symmetric $n \times n$ matrices (matrix representation of second derivatives of $C^2$ functions, quadratic forms...). e.g. second order conditions in optimization, checking concavity and convexity, Taylor series approximation of function

- Recall that an $n \times n$ real matrix $A$ is *symmetric* if $a_{ij} = a_{ji}$ for all $i, j$, where $a_{ij}$ is the $(i, j)^{\text{th}}$ entry of $A$.

Rest of this section: work in $\mathbb{R}^n$:
- vector space
- norm
- inner product $\left(x \cdot y = \sum_{i=1}^{n} x_i y_i\right)$

# Orthonormal Bases

**Definition 1.** *Let*

$$\delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

*A basis* $V = \{v_1, \ldots, v_n\}$ *of* $\mathbf{R}^n$ *is* orthonormal *if* $v_i \cdot v_j = \delta_{ij}$. $= \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$

In other words, a basis is orthonormal if each basis element has unit length ( $\|v_i\|^2 = v_i \cdot v_i = 1 \ \forall i$), and distinct basis elements are perpendicular ($v_i \cdot v_j = 0$ for $i \neq j$).

# Orthonormal Bases

**Remark**: Suppose that $x = \sum_{j=1}^{n} \alpha_j v_j$ where $\{v_1, \ldots, v_n\}$ is an orthonormal basis of $\mathbf{R}^n$. Then

$$\begin{aligned} x \cdot v_k &= \left( \sum_{j=1}^{n} \alpha_j v_j \right) \cdot v_k \\ &= \sum_{j=1}^{n} \alpha_j (v_j \cdot v_k) \\ &= \sum_{j=1}^{n} \alpha_j \delta_{jk} \quad = \begin{cases} 1 & j = k \\ 0 & j \neq k \end{cases} \\ &= \alpha_k \end{aligned}$$

so

$$x = \sum_{j=1}^{n} (x \cdot v_j) v_j$$

# Orthonormal Bases

**Example:** The standard basis of $\mathbf{R}^n$ is orthonormal.

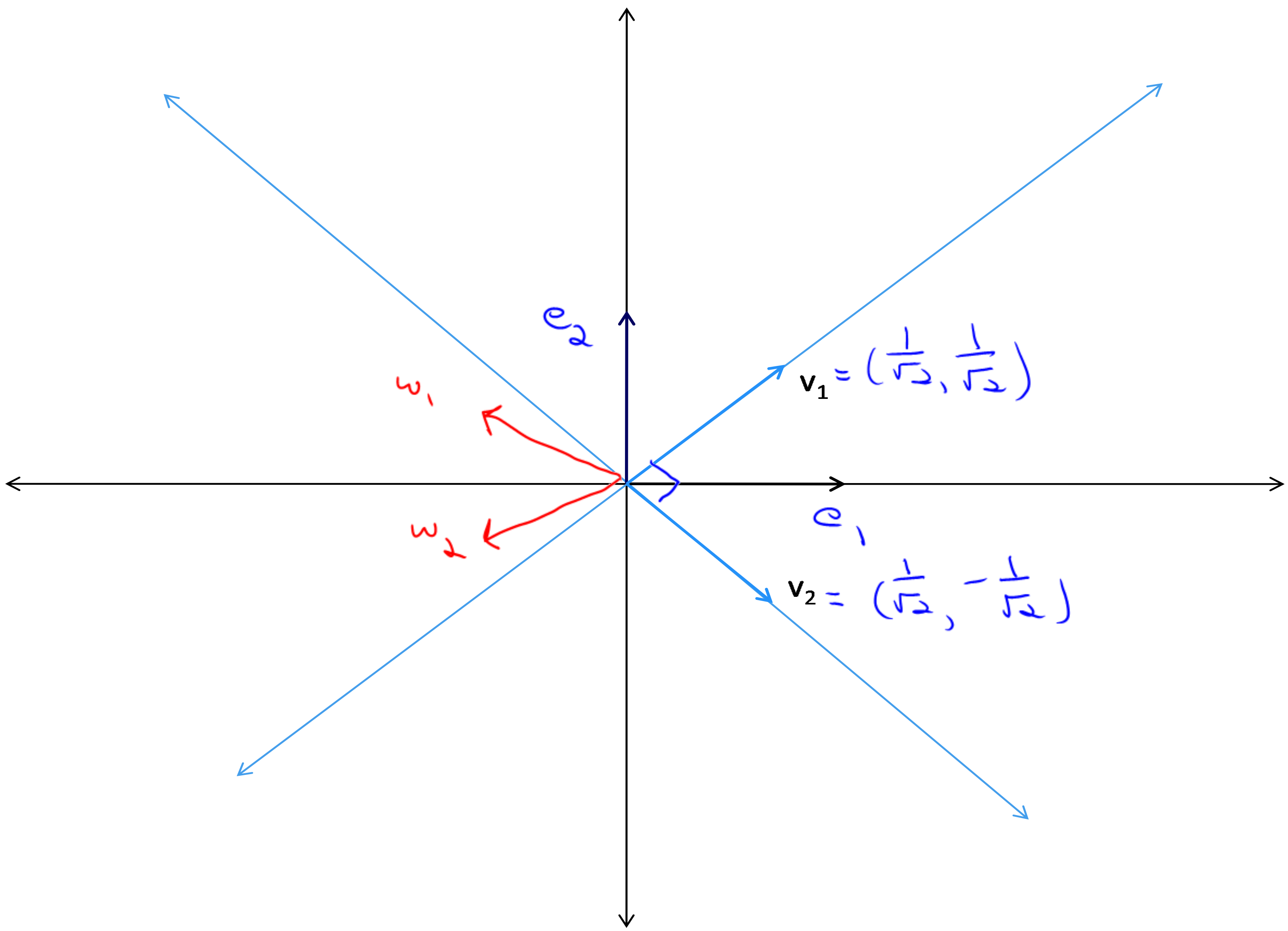
$e_i = (0, \dots, 1, 0, \dots, 0) \qquad i = 1, \dots, n$

(Why?)

e.g. $\mathbb{R}^2$: $e_1 = (1, 0)$ , $e_2 = (0, 1)$

others? e.g. $v_1 = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$ , $v_2 = \left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$

also many bases that are <u>not</u> orthonormal


$e_2$
$w_1$
$\mathbf{v}_1 = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$
$e_1$
$w_2$
$\mathbf{v}_2 = \left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$

# Unitary Matrices

Recall that for a real $n \times m$ matrix $A$, $A^\top$ denotes the transpose of $A$: the $(i,j)^{th}$ entry of $A^\top$ is the $(j,i)^{th}$ entry of $A$.

So the $i^{th}$ row of $A^\top$ is the $i^{th}$ column of $A$.

**Definition 2.** *A real $n \times n$ matrix $A$ is* unitary *if $A^\top = A^{-1}$.*

Notice that by definition every unitary matrix is invertible.

# Unitary Matrices

**Theorem 1.** *A real $n \times n$ matrix $A$ is unitary if and only if the columns of $A$ are orthonormal.*

*Proof.* Let $v_j$ denote the $j^{th}$ column of $A$.

$$\begin{aligned} A^\top = A^{-1} \quad &\iff \quad A^\top A = I \; = (\delta_{ij}) \begin{cases} 1 & i=j \\ 0 & i \neq j \end{cases} \\ &\iff \quad v_i \cdot v_j = \delta_{ij} \quad \forall i, j \\ &\iff \quad \{v_1, \ldots, v_n\} \text{ is orthonormal} \end{aligned}$$

$\square$

$A = (v_1 \; \cdots\cdots \; v_n)$ - change of basis from V to standard basis in $\mathbb{R}^n$ (W)

$\text{ord}_W(v_1) \; \cdots \; \text{ord}_W(v_n)$

# Unitary Matrices

If $A$ is unitary, let $V$ be the set of columns of $A$ and $W$ be the standard basis of $\mathbf{R}^n$. Since $A$ is unitary, it is invertible, so $V$ is a basis of $\mathbf{R}^n$. ( $\{v_1, \ldots, v_n\}$ linearly independent )

$$A^\top = A^{-1} = Mtx_{V,W}(id)$$ = change of basis from W to V

standard basis

Since $V$ is orthonormal, the transformation between bases $W$ and $V$ preserves all geometry, including lengths and angles.

<u>Thm</u> : Let $C$ be an $n \times n$ real symmetric matrix. Then $C$ is diagonalizable. In addition,

$$C = P^{-1} D P$$

where $D$ is a diagonal matrix and $P$ is unitary.

Note : • The diagonal elements $\{\lambda_1, \ldots, \lambda_n\}$ of $D$ are the eigenvalues of $C$

• $C$ has orthonormal eigenvectors $\{v_1, \ldots, v_n\}$ that are a basis for $\mathbb{R}^n$.

# Diagonalization of Real Symmetric Matrices

**Theorem 2.** *Let $T \in L(\mathbf{R}^n, \mathbf{R}^n)$ and $W$ be the standard basis of $\mathbf{R}^n$. Suppose that $\overset{=C}{Mtx_W(T)}$ is symmetric. Then the eigenvectors of $T$ are all real, and there is an orthonormal basis $V = \{v_1, \ldots, v_n\}$ of $\mathbf{R}^n$ consisting of eigenvectors of $T$, so that $Mtx_W(T)$ is diagonalizable:*

$$C = Mtx_W(T) = \overbrace{Mtx_{W,V}(id)}^{\text{unitary}} \cdot \overbrace{Mtx_V(T)}^{\text{diagonal}} \cdot \overbrace{Mtx_{V,W}(id)}^{\text{unitary}}$$

*where $Mtx_V T$ is diagonal and the change of basis matrices $Mtx_{V,W}(id)$ and $Mtx_{W,V}(id)$ are unitary.*

The proof of the theorem requires a lengthy digression into the linear algebra of *complex* vector spaces. A brief outline is in the notes.

# Quadratic Forms

**Example:** Let $f : \mathbb{R}^2 \to \mathbb{R}$

$$f(x) = \alpha x_1^2 + \beta x_1 x_2 + \gamma x_2^2$$

write as $f(x) = x^T A x$, $A$ symmetric

Let

$$A = \begin{pmatrix} \alpha & \frac{\beta}{2} \\ \frac{\beta}{2} & \gamma \end{pmatrix}$$

$$x^T A x = (x_1 \quad x_2) \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

so $A$ is symmetric and

$$
\begin{aligned}
x^\top A x = (x_1, x_2) \begin{pmatrix} \alpha & \frac{\beta}{2} \\ \frac{\beta}{2} & \gamma \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}
&= (x_1, x_2) \begin{pmatrix} \alpha x_1 + \frac{\beta}{2} x_2 \\ \frac{\beta}{2} x_1 + \gamma x_2 \end{pmatrix} \\
&= \alpha x_1^2 + \beta x_1 x_2 + \gamma x_2^2 \\
&= f(x)
\end{aligned}
$$

Notice $f(0) = 0$.

Can we determine anything about $f(x)$ for $x \neq 0$ ?

e.g. $f(x) \geq 0 \quad \forall x$ ?

easy if $\beta = 0$ ..

# Quadratic Forms

general form:

Consider a quadratic form :

$$f(x_1, \ldots, x_n) = \sum_{i=1}^{n} \alpha_{ii} x_i^2 + \sum_{i<j} \beta_{ij} x_i x_j \tag{1}$$

Let

$$\alpha_{ij} = \begin{cases} \frac{\beta_{ij}}{2} & \text{if } i < j \\ \frac{\beta_{ji}}{2} & \text{if } i > j \end{cases}$$

above diagonal

below diagonal

Let

$$A = \begin{pmatrix} \alpha_{11} & \cdots & \alpha_{1n} \\ \vdots & \ddots & \vdots \\ \alpha_{n1} & \cdots & \alpha_{nn} \end{pmatrix} \text{ so } f(x) = x^\top A x$$

real symmetric

# Quadratic Forms

$A$ is symmetric, so let $V = \{v_1, \ldots, v_n\}$ be an orthonormal basis of eigenvectors of $A$ with corresponding eigenvalues $\lambda_1, \ldots, \lambda_n$.

$$
\begin{aligned}
\text{Then } A &= U^\top D U \quad = U^{-1} D U \\
\text{where } D &= \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{pmatrix} \\
\text{and } U &= Mtx_{V,W}(id) \text{ is unitary}
\end{aligned}
$$

$U^{-1} =$

The columns of $U^\top$ (the rows of $U$) are the coordinates of $v_1, \ldots, v_n$, expressed in terms of the standard basis $W$. Given $x \in \mathbf{R}^n$, recall

$$x = \sum_{i=1}^{n} \gamma_i v_i \text{ where } \gamma_i = x \cdot v_i$$

# Quadratic Forms

So

$$
\begin{aligned}
f(x) &= f\left(\sum \gamma_i v_i\right) \\
&= \left(\sum \gamma_i v_i\right)^\top A \left(\sum \gamma_i v_i\right) \quad = x^\top A x \\
&= \left(\sum \gamma_i v_i\right)^\top U^\top D U \left(\sum \gamma_i v_i\right) \\
&= \left(U \sum \gamma_i v_i\right)^\top D \left(U \sum \gamma_i v_i\right) \quad \left( (EF)^\top = F^\top E^\top \right) \\
&= \left(\sum \gamma_i U v_i\right)^\top D \left(\sum \gamma_i U v_i\right) \quad (U \text{ linear}) \\
&= (\gamma_1, \ldots, \gamma_n) D \begin{pmatrix} \gamma_1 \\ \vdots \\ \gamma_n \end{pmatrix} \quad \leftarrow \\
&= \sum \lambda_i \gamma_i^2
\end{aligned}
$$

$U$ is change of basis from $W$ to $V$ $\Rightarrow$

$\forall i \; U v_i = e_i = (0, \ldots, 1, 0, \ldots, 0)$

$\mathrm{crd}_V(v_i)$

$\uparrow$ eigenvalues of $A$

# Quadratic Forms

The equation for a level set of $f$ is

$$\{\gamma \in \mathbb{R}^n : f(\gamma) = C\} = \left\{\gamma \in \mathbf{R}^n : \sum_{i=1}^{n} \lambda_i \gamma_i^2 = C\right\} \qquad C \in \mathbb{R}$$

- $\boxed{\text{If } \lambda_i \geq 0 \text{ for all } i,}$ the level set is an ellipsoid, with principal axes in the directions $v_1, \ldots, v_n$. The length of the principal axis along $v_i$ is $\sqrt{C/\lambda_i}$ if $C \geq 0$ (if $\lambda_i = 0$, the level set is a degenerate ellipsoid with principal axis of infinite length in that direction). The level set is empty if $C < 0$.

  $\Rightarrow$ $f$ has global min at 0, $f(x) \geq 0 \quad \forall x$

- $\boxed{\text{If } \lambda_i \leq 0 \text{ for all } i,}$ the level set is an ellipsoid, with principal axes in the directions $v_1, \ldots, v_n$. The length of the principal

  $\Rightarrow$ $f$ has global max at 0, $f(x) \leq 0 \quad \forall x$

axis along $v_i$ is $\sqrt{|C|/\lambda_i}$ if $C \leq 0$ (if $\lambda_i = 0$, the level set is a degenerate ellipsoid with principal axis of infinite length in that direction). The level set is empty if $C > 0$.

- If $\lambda_i > 0$ for some $i$ and $\lambda_j < 0$ for some $j$, the level set is a hyperboloid. For example, suppose $n = 2$, $\lambda_1 > 0$, $\lambda_2 < 0$. The equation is

$$
\begin{aligned}
C &= \lambda_1 \gamma_1^2 + \lambda_2 \gamma_2^2 \\
&= \left(\sqrt{\lambda_1}\gamma_1 + \sqrt{|\lambda_2|}\gamma_2\right)\left(\sqrt{\lambda_1}\gamma_1 - \sqrt{|\lambda_2|}\gamma_2\right)
\end{aligned}
$$

$\Rightarrow$ $f$ has a saddle point at $0$

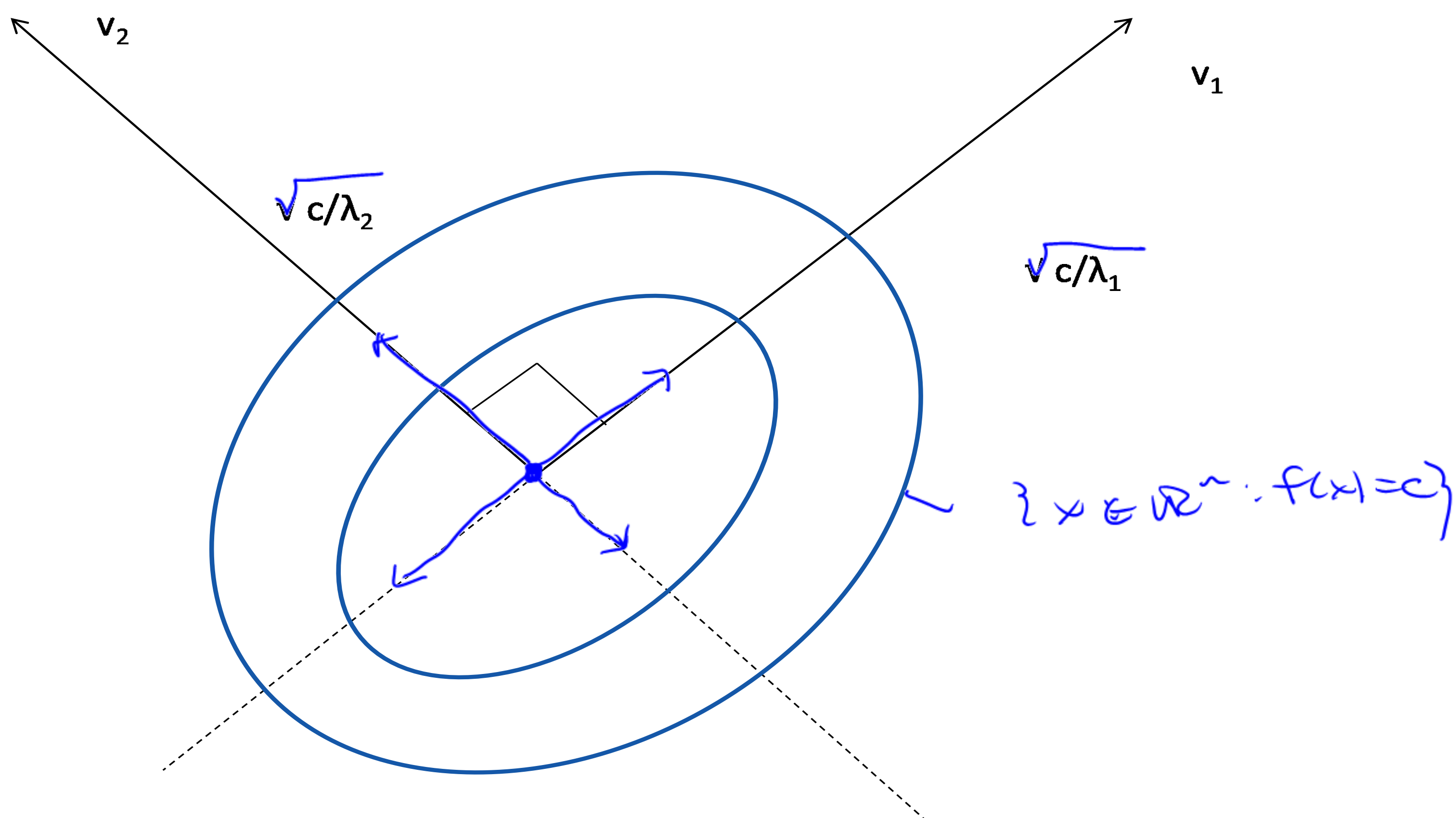
min with respect to $v_i$

max with respect to $v_j$

This is a hyperbola with asymptotes

$$\begin{aligned} 0 &= \sqrt{\lambda_1}\gamma_1 + \sqrt{|\lambda_2|}\gamma_2 \\ \Rightarrow \gamma_1 &= -\sqrt{\frac{|\lambda_2|}{\lambda_1}}\gamma_2 \\ 0 &= \left(\sqrt{\lambda_1}\gamma_1 - \sqrt{|\lambda_2|}\gamma_2\right) \\ \Rightarrow \gamma_1 &= \sqrt{\frac{|\lambda_2|}{\lambda_1}}\gamma_2 \end{aligned}$$

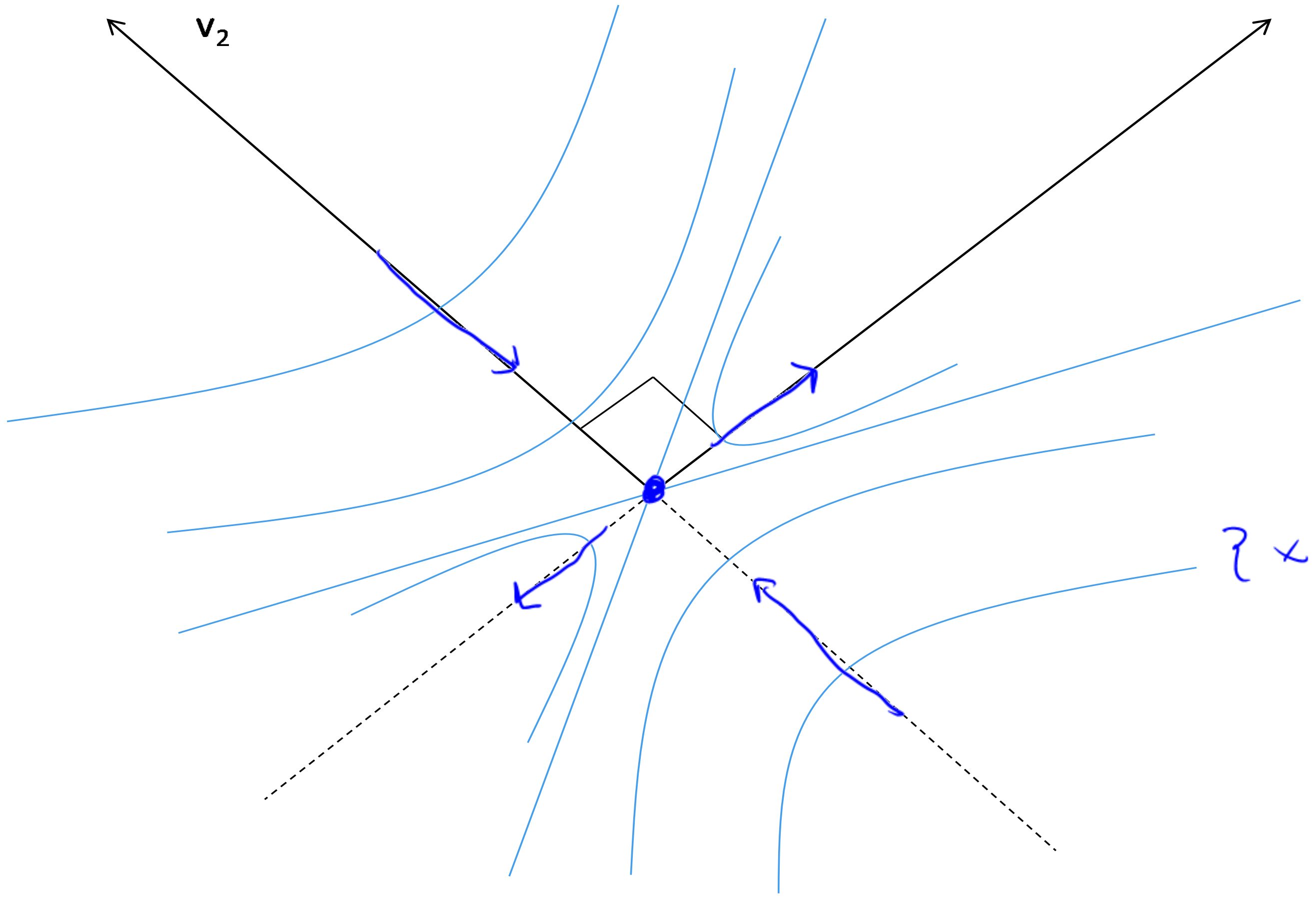
$\lambda_1 > 0, \lambda_2 > 0$




f has a global min at 0

$\lambda_1 > 0, \lambda_2 < 0$

$\gamma_1 = \sqrt{|\lambda_2|/\lambda_1}\ \gamma_2$




$\gamma_1 = -\sqrt{\frac{|\lambda_2|}{\lambda_1}}\ \gamma_2$

$\{x \in \mathbb{R}^n : f(x) = c\}$

f has a saddle point at 0

# Quadratic Forms

This proves the following corollary of Theorem 2.

**Corollary 1.** *Consider the quadratic form (1).* Let $\{v_1, \ldots, v_n\}$ be an orthonormal basis of eigenvectors of $A$ with corresponding eigenvalues $\{\lambda_1, \ldots, \lambda_n\}$.

1. *$f$ has a global minimum at 0 if and only if $\lambda_i \geq 0$ for all $i$; the level sets of $f$ are ellipsoids with principal axes aligned with the orthonormal eigenvectors $v_1, \ldots, v_n$.*

2. *$f$ has a global maximum at 0 if and only if $\lambda_i \leq 0$ for all $i$; the level sets of $f$ are ellipsoids with principal axes aligned with the orthonormal eigenvectors $v_1, \ldots, v_n$.*

3. *If* $\lambda_i < 0$ *for some* $i$ *and* $\lambda_j > 0$ *for some* $j$, *then* $f$ *has a saddle point at 0; the level sets of* $f$ *are hyperboloids with principal axes aligned with the orthonormal eigenvectors* $v_1, \ldots, v_n$.

# Bounded Linear Maps

over $\mathbb{R}$

**Definition 3.** *Suppose $X, Y$ are normed vector spaces and $T \in L(X, Y)$. We say $T$ is* bounded *if*

$$\exists \beta \in \mathbf{R} \text{ s.t. } \|T(x)\|_Y \leq \beta \|x\|_X \quad \forall x \in X$$

Note this implies that $T$ is Lipschitz with constant $\beta$.

why not previous notion of bounded:

$$\exists B \in \mathbb{R} \text{ s.t. } \|T(x)\| \leq B \quad \forall x \text{ ??}$$

$$T(\alpha x) = \alpha T(x) \quad \forall \alpha \in \mathbb{R}$$

$$\Rightarrow \quad \|T(\alpha x)\| = |\alpha| \, \|T(x)\| \quad \forall \alpha \in \mathbb{R}$$

# Bounded Linear Maps

Much more is true:

**Theorem 3** (Thms. 4.1, 4.3). *Let $X$ and $Y$ be normed vector spaces and $T \in L(X, Y)$. Then*

$$\begin{aligned} & T \text{ is continuous at some point } x_0 \in X \\ \iff \quad & T \text{ is continuous at every } x \in X \\ \iff \quad & T \text{ is uniformly continuous on } X \\ \iff \quad & T \text{ is Lipschitz} \\ \iff \quad & T \text{ is bounded} \end{aligned}$$

*Proof.* Suppose $T$ is continuous at $x_0$. Fix $\varepsilon > 0$. Then there exists $\delta > 0$ such that

$$\|z - x_0\| < \delta \Rightarrow \|T(z) - T(x_0)\| < \varepsilon$$

Now suppose $x$ is any element of $X$. If $\|y - x\| < \delta$, let $z =$ $\underbrace{z = y - x + x_0}_{z - x_0 = y - x}$, so $\|z - x_0\| = \|y - x\| < \delta$.

$$
\begin{aligned}
&\|T(y) - T(x)\| \\
&\quad = \|T(y - x)\| && \text{(T linear)} \\
&\quad = \|T(y - x + x_0 - x_0))\| && = \|T(z - x_0)\| \\
&\quad = \|T(z) - T(x_0)\| && \text{(T linear)} \\
&\quad < \varepsilon
\end{aligned}
$$

which proves that $T$ is continuous at every $x$, and uniformly continuous.

We claim that $T$ is bounded if and only if $T$ is continuous at 0. Suppose $T$ is not bounded. Then

$$\exists \{x_n\} \text{ s.t. } \|T(x_n)\| > n\|x_n\| \quad \forall n$$

Note that $x_n \neq 0$. Let $\varepsilon = 1$. Fix $\delta > 0$ and choose $n$ such that $\frac{1}{n} < \delta$. Let

$$
\begin{aligned}
x'_n &= \frac{x_n}{n\|x_n\|} \quad = \frac{1}{n} \frac{x_n}{\|x_n\|} \\
\|x'_n - 0\| = \|x'_n\| &= \frac{\|x_n\|}{n\|x_n\|} \\
&= \frac{1}{n} \\
&< \delta \\
\|T(x'_n) - T(0)\| &= \|T(x'_n)\| \\
&= \frac{1}{n\|x_n\|}\|T(x_n)\| \quad (\text{defn of } x'_n + T \text{ linear}) \\
&> \frac{n\|x_n\|}{n\|x_n\|} \quad (\text{defn of } x_n) \\
&= 1 \\
&= \varepsilon
\end{aligned}
$$

$T(x'_n) = T\left(\frac{x_n}{n\|x_n\|}\right) = \frac{1}{n\|x_n\|} T(x_n)$

$T(0) = 0$

Since this is true for every $\delta$, $T$ is not continuous at 0. Therefore, $T$ continuous at 0 implies $T$ is bounded. Now, suppose $T$ is bounded, so find $M$ $>0$ such that $\|T(x)\| \leq M\|x\|$ for every $x \in X$. Given $\varepsilon > 0$, let $\delta = \varepsilon/M$. Then

$$\begin{aligned}
\|x - 0\| < \delta \quad &\Rightarrow \quad \|x\| < \delta \\
&\Rightarrow \quad \|T(x) - T(0)\| = \|T(x)\| < M\delta \qquad \text{(defn of } M\text{)} \\
&\Rightarrow \quad \|T(x) - T(0)\| < \varepsilon = M\delta
\end{aligned}$$

so $T$ is continuous at 0.

$\|T(x)\| \leq M\|x\| < M\delta$

since $\|x\| < \delta$

Thus, we have shown that continuity at some point $x_0$ implies uniform continuity, which implies continuity at every point, which implies $T$ is continuous at 0, which implies that $T$ is bounded, which implies that $T$ is continuous at 0, which implies that $T$ is continuous at some $x_0$, so all of the statements except possibly the Lipschitz statement are equivalent.

Suppose $T$ is bounded, with constant $M$ $>0$. Then

$$\begin{aligned}\|T(x)-T(y)\| &= \|T(x-y)\| \qquad \text{(T linear)}\\ &\leq M\|x-y\|\end{aligned}$$

so $T$ is Lipschitz with constant $M$; conversely, if $T$ is Lipschitz with constant $M$, then $T$ is bounded with constant $M$. So all the statements are equivalent. $\square$

$$\forall x \in X: \quad \|T(x)-T(0)\| = \|T(x)\| \leq M\|x-0\| = M\|x\|$$

# Bounded Linear Maps

Every linear map on a finite-dimensional normed vector space is bounded (and thus continuous, uniformly continuous, and Lipschitz continuous).

**Theorem 4** (Thm. 4.5)**.** *Let $X$ and $Y$ be normed vector spaces, with* $\dim X = n$. *Every $T \in L(X,Y)$ is bounded.*

$n \in \mathbb{N}$

*Proof.* See de la Fuente. $\square$

# Topological Isomorphism

**Definition 4.** *A* topological isomorphism *between normed vector spaces $X$ and $Y$ is a linear transformation $T \in L(X,Y)$ that is invertible (one-to-one, onto), continuous, and has a continuous inverse.*

*Two normed vector spaces $X$ and $Y$ are* topologically isomorphic *if there is a topological isomorphism $T : X \to Y$.*

# The Space $B(X,Y)$

$\exists \beta > 0$ s.t.
$\|T(x)\| \leq \beta \|x\|$
$\forall x$

$\Rightarrow$

Suppose $X$ and $Y$ are normed vector spaces. We define

$$
\begin{aligned}
B(X,Y) &= \{T \in L(X,Y) : T \text{ is bounded}\} \\
\|T\|_{B(X,Y)} &= \sup\left\{\frac{\|T(x)\|_Y}{\|x\|_X}, x \in X, x \neq 0\right\} \\
&= \sup\{\|T(x)\|_Y : \|x\|_X = 1\}
\end{aligned}
$$

$$\frac{\|T(x)\|}{\|x\|} \leq \beta \quad \forall x \neq 0$$

$\Rightarrow \quad \|T(x)\| \leq \|T\| \|x\|$ by defn.

We skip the proofs of the rest of these results – read dIF.

$$
\begin{aligned}
\frac{\|T(x)\|}{\|x\|} &= \frac{1}{\|x\|} \|T(x)\| \\
&= \left\| \frac{1}{\|x\|} T(x) \right\| \\
&= \left\| T\left(\frac{x}{\|x\|}\right) \right\|
\end{aligned}
$$

# The Space $B(X, Y)$

**Theorem 5** (Thm. 4.8)**.** *Let $X, Y$ be normed vector spaces. Then*

$$\Big(B(X,Y), \|\cdot\|_{B(X,Y)}\Big)$$

*is a normed vector space.*

# The Space $B(\mathbf{R}^n, \mathbf{R}^m)$

**Theorem 6** (Thm. 4.9)**.** *Let* $T \in L(\mathbf{R}^n, \mathbf{R}^m)$ *(*$= B(\mathbf{R}^n, \mathbf{R}^m)$*)* *with matrix* $A = (a_{ij})$ *with respect to the standard bases. Let*

$$M = \max\{|a_{ij}| : 1 \le i \le m, 1 \le j \le n\}$$

*Then*

$$M \le \|T\| \le M\sqrt{mn}$$

.

# Compositions

**Theorem 7** (Thm. 4.10)**.** *Let* $R \in L(\mathbf{R}^m, \mathbf{R}^n)$ *and* $S \in L(\mathbf{R}^n, \mathbf{R}^p)$. *Then*

$$\|S \circ R\| \leq \|S\| \|R\|$$

# Invertibility

Define $\Omega(\mathbf{R}^n) = \{T \in L(\mathbf{R}^n, \mathbf{R}^n) : T \text{ is invertible}\}$

**Theorem 8** (Thm. 4.11'). *Suppose $T \in L(\mathbf{R}^n, \mathbf{R}^n)$ and $E$ is the standard basis of $\mathbf{R}^n$. Then*

$$\begin{aligned} & T \text{ is invertible} \\ \iff \quad & \ker T = \{0\} \\ \iff \quad & \det\left(Mtx_E(T)\right) \neq 0 \\ \iff \quad & \det\left(Mtx_{V,V}(T)\right) \neq 0 \text{ for every basis } V \\ \iff \quad & \det\left(Mtx_{V,W}(T)\right) \neq 0 \text{ for every pair of bases } V, W \end{aligned}$$

# Invertibility

**Theorem 9** (Thm. 4.12)**.** *If* $S, T \in \Omega(\mathbf{R}^n)$, *then* $S \circ T \in \Omega(\mathbf{R}^n)$ *and*

$$(S \circ T)^{-1} = T^{-1} \circ S^{-1}$$

# Invertibility

**Theorem 10** (Thm. 4.14)**.** *Let* $S, T \in L(\mathbf{R}^n, \mathbf{R}^n)$*. If* $T$ *is invertible and*

$$\|T - S\| < \frac{1}{\|T^{-1}\|}$$

*then* $S$ *is invertible. In particular,* $\Omega(\mathbf{R}^n)$ *is open in* $L(\mathbf{R}^n, \mathbf{R}^n) = B(\mathbf{R}^n, \mathbf{R}^n)$*.*

**Theorem 11** (Thm. 4.15)**.** *The function* $(\cdot)^{-1} : \Omega(\mathbf{R}^n) \to \Omega(\mathbf{R}^n)$ *that assigns* $T^{-1}$ *to each* $T \in \Omega(\mathbf{R}^n)$ *is continuous.*