Econ 204 2018

Lecture 10

Outline

1. Diagonalization of Real Symmetric Matrices
2. Application to Quadratic Forms
3. Linear Maps Between Normed Spaces

Announcements:
- PS 3 due now
  → solutions × 2 pm today
- PS 4 posted
  → due Tuesday
- last year’s exam packed × Sunday
How Might This Matter

- Why does diagonalizability matter?

Consider a two-dimensional linear difference equation:

\[
\begin{pmatrix} c_{t+1} \\ k_{t+1} \end{pmatrix} = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} \begin{pmatrix} c_t \\ k_t \end{pmatrix} \quad \forall t = 0, 1, 2, 3, \ldots
\]

given an initial condition \(c_0, k_0\), or, setting

\[
y_t = \begin{pmatrix} c_t \\ k_t \end{pmatrix} \quad \forall t \quad \text{and} \quad B = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}
\]

we can rewrite this more compactly as

\[
y_{t+1} = By_t \quad \forall t
\]

where \(b_{ij} \in \mathbb{R}\) each \(i, j\).
We want to find a solution $y_t$, $t = 1, 2, 3, \ldots$ given initial condition $y_0$. (Why?)

Such a dynamical system will arise for example as a characterization of the solution to a standard infinite-horizon optimal growth problem (202a, lecture 2).

If $B$ is diagonalizable, this can be easily solved after a change of basis. If $B$ is diagonalizable, choose an invertible $2 \times 2$ real matrix $P$ such that

$$P^{-1}BP = D = \begin{pmatrix} d_1 & 0 \\ 0 & d_2 \end{pmatrix}$$

Then

$$y_{t+1} = By_t \quad \forall t \iff P^{-1}y_{t+1} = P^{-1}By_t \quad \forall t \quad \text{(multiplied by $P^{-1}$)}$$

$$\iff \begin{pmatrix} P^{-1}y_{t+1} \end{pmatrix} = \begin{pmatrix} P^{-1}BP \end{pmatrix} \begin{pmatrix} P^{-1}y_t \end{pmatrix} \quad \forall t \quad pp^{-1} = I$$

$$\iff \bar{y}_{t+1} = D\bar{y}_t \quad \forall t$$

$$= \begin{pmatrix} d_1 & 0 \\ 0 & d_2 \end{pmatrix} \begin{pmatrix} \bar{y}_t \end{pmatrix}$$

where $\bar{y}_t = P^{-1}y_t \quad \forall t$
where $\bar{y}_t = P^{-1}y_t \ \forall t$.

Since $D$ is diagonal, after a change of basis to $\bar{y}_t$, we need to solve two independent linear univariate difference equations, which is easy:

$$\bar{y}_{it} = d_{it}\bar{y}_{i0} \ \forall t$$

- Not all real $n \times n$ matrices are diagonalizable (not even all invertible $n \times n$ matrices are)...so can we identify some classes that are? 
  - basis of eigenvectors ($\iff$) 
  - $n$ distinct eigenvalues ($\iff$)

- Some types of matrices appear more frequently than others – especially real symmetric $n \times n$ matrices (matrix representation of second derivatives of $C^2$ functions, quadratic forms...).
  - e.g. second order conditions in optimization, checking concavity and convexity, Taylor series approximation of function
• Recall that an $n \times n$ real matrix $A$ is symmetric if $a_{ij} = a_{ji}$ for all $i, j$, where $a_{ij}$ is the $(i, j)^{th}$ entry of $A$. 
Orthonormal Bases

Definition 1. Let

\[ \delta_{ij} = \begin{cases} 
1 & \text{if } i = j \\
0 & \text{if } i \neq j 
\end{cases} \]

A basis \( V = \{v_1, \ldots, v_n\} \) of \( \mathbb{R}^n \) is orthonormal if \( v_i \cdot v_j = \delta_{ij} \).

In other words, a basis is orthonormal if each basis element has unit length \( (\|v_i\|^2 = v_i \cdot v_i = 1 \ \forall i) \), and distinct basis elements are perpendicular \( (v_i \cdot v_j = 0 \text{ for } i \neq j) \).
**Orthonormal Bases**

**Remark:** Suppose that \( x = \sum_{j=1}^{n} \alpha_j v_j \) where \( \{v_1, \ldots, v_n\} \) is an orthonormal basis of \( \mathbb{R}^n \). Then

\[
x \cdot v_k = \left( \sum_{j=1}^{n} \alpha_j v_j \right) \cdot v_k \\
= \sum_{j=1}^{n} \alpha_j (v_j \cdot v_k) \\
= \sum_{j=1}^{n} \alpha_j \delta_{jk} = \sum_{j=1}^{n} \alpha_j \delta_{jk}^{s=k} = \delta_{k}^{s=k} = \alpha_k
\]

so

\[
x = \sum_{j=1}^{n} (x \cdot v_j) v_j
\]
Example: The standard basis of $\mathbb{R}^n$ is orthonormal.

$e_i = (0, \ldots, 1, 0, \ldots, 0) \quad i = 1, \ldots, n$

(Why?)

e.g., $\mathbb{R}^2$: $e_1 = (1, 0)$, $e_2 = (0, 1)$

others? e.g., $v_1 = (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$, $v_2 = (\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}})$

also many bases that are not orthonormal
\[ \mathbf{e}_2 = (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}) \]

\[ \mathbf{e}_1 = (\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}) \]

\[ \mathbf{v}_1 = (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}) \]

\[ \mathbf{v}_2 = (\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}) \]
Unitary Matrices

Recall that for a real $n \times m$ matrix $A$, $A^\top$ denotes the transpose of $A$: the $(i,j)^{th}$ entry of $A^\top$ is the $(j,i)^{th}$ entry of $A$.

So the $i^{th}$ row of $A^\top$ is the $i^{th}$ column of $A$.

**Definition 2.** A real $n \times n$ matrix $A$ is unitary if $A^\top = A^{-1}$.

Notice that by definition every unitary matrix is invertible.
Unitary Matrices

Theorem 1. A real $n \times n$ matrix $A$ is unitary if and only if the columns of $A$ are orthonormal.

Proof. Let $v_j$ denote the $j^{th}$ column of $A$.

\[
A^\top = A^{-1} \iff A^\top A = I = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} \begin{pmatrix} 1 & \cdots & 1 \end{pmatrix} \\
\iff v_i \cdot v_j = \delta_{ij} \quad \forall i, j \\
\iff \{v_1, \ldots, v_n\} \text{ is orthonormal} 
\]

\[\square\]
If $A$ is unitary, let $V$ be the set of columns of $A$ and $W$ be the standard basis of $\mathbb{R}^n$. Since $A$ is unitary, it is invertible, so $V$ is a basis of $\mathbb{R}^n$. \( \{v_1, \ldots, v_n\} \) linearly independent

\[
A^\top = A^{-1} = Mtx_{V,W}(id) = \text{change of basis from } W \text{ to } V
\]

Since $V$ is orthonormal, the transformation between bases $W$ and $V$ preserves all geometry, including lengths and angles.
Thus: Let $C$ be an $n \times n$ real symmetric matrix. Then $C$ is diagonalizable. In addition,

$$C = P^{-1}DP$$

where $D$ is a diagonal matrix and $P$ is unitary.

Note: The diagonal elements $d_1, \ldots, d_n$ of $D$ are the eigenvalues of $C$.

- $C$ has orthonormal eigenvectors $v_1, \ldots, v_n$ that are a basis for $\mathbb{R}^n$. 
Diagonalization of Real Symmetric Matrices

**Theorem 2.** Let $T \in L(\mathbb{R}^n, \mathbb{R}^n)$ and $W$ be the standard basis of $\mathbb{R}^n$. Suppose that $Mtx_W(T)$ is symmetric. Then the eigenvectors of $T$ are all real, and there is an orthonormal basis $V = \{v_1, \ldots, v_n\}$ of $\mathbb{R}^n$ consisting of eigenvectors of $T$, so that $Mtx_W(T)$ is diagonalizable:

$$Mtx_W(T) = Mtx_W,V(id) \cdot Mtx_V(T) \cdot Mtx_V,W(id)$$

where $Mtx_V T$ is diagonal and the change of basis matrices $Mtx_{V,W}(id)$ and $Mtx_{W,V}(id)$ are unitary.

The proof of the theorem requires a lengthy digression into the linear algebra of complex vector spaces. A brief outline is in the notes.
Quadratic Forms

Example: Let \( f : \mathbb{R}^2 \to \mathbb{R} \)

\[
f(x) = \alpha x_1^2 + \beta x_1 x_2 + \gamma x_2^2
\]

Let \( A \) write as \( f(x) = x^T A x \), \( A \) symmetric

\[
A = \begin{pmatrix}
\alpha & \beta \\
\beta & 2
\end{pmatrix}
\]

\[
x^T A x = (x_1, x_2) \begin{pmatrix}
\alpha & \beta \\
\beta & 2
\end{pmatrix} \begin{pmatrix}
x_1 \\
x_2
\end{pmatrix}
\]
so $A$ is symmetric and

$$x^\top A x = (x_1, x_2) \begin{pmatrix} \alpha & \frac{\beta}{2} \\ \frac{\beta}{2} & \gamma \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$= (x_1, x_2) \begin{pmatrix} \alpha x_1 + \frac{\beta}{2} x_2 \\ \frac{\beta}{2} x_1 + \gamma x_2 \end{pmatrix}$$

$$= \alpha x_1^2 + \beta x_1 x_2 + \gamma x_2^2$$

$$= f(x)$$

Notice $f(0) = 0$.

Can we determine anything about $f(x)$ for $x \neq 0$?

e.g. $f(x) > 0 \ \forall x$? easy if $\beta = 0$..
Quadratic Forms

Consider a quadratic form:

\[ f(x_1, \ldots, x_n) = \sum_{i=1}^{n} \alpha_{ii}x_i^2 + \sum_{i<j} \beta_{ij}x_ix_j \]  

Let

\[ \alpha_{ij} = \begin{cases} \frac{\beta_{ij}}{2} & \text{if } i < j \\ \frac{\beta_{ii}}{2} & \text{if } i > j \end{cases} \]  

Let

\[ A = \begin{pmatrix} \alpha_{11} & \cdots & \alpha_{1n} \\ \vdots & \ddots & \vdots \\ \alpha_{n1} & \cdots & \alpha_{nn} \end{pmatrix} \]  

so \( f(x) = x^\top Ax \)
Quadratic Forms

$A$ is symmetric, so let $V = \{v_1, \ldots, v_n\}$ be an orthonormal basis of eigenvectors of $A$ with corresponding eigenvalues $\lambda_1, \ldots, \lambda_n$.

Then $A = U^\top DU = U^\top \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{pmatrix} U$

where $D = \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{pmatrix}$

and $U = Mtx_{V,W}(id)$ is unitary

The columns of $U^\top$ (the rows of $U$) are the coordinates of $v_1, \ldots, v_n$, expressed in terms of the standard basis $W$. Given $x \in \mathbb{R}^n$, recall

$$x = \sum_{i=1}^n \gamma_i v_i \text{ where } \gamma_i = x \cdot v_i$$
Quadratic Forms

So

\[ f(x) = f \left( \sum \gamma_i v_i \right) \]
\[ = \left( \sum \gamma_i v_i \right)^T A \left( \sum \gamma_i v_i \right) = \sum \lambda_i \gamma_i^2 \]
\[ = \left( \sum \gamma_i v_i \right)^T U^T DU \left( \sum \gamma_i v_i \right) \]
\[ = \left( U \sum \gamma_i v_i \right)^T D \left( U \sum \gamma_i v_i \right) \]
\[ = \left( \sum \gamma_i U v_i \right)^T D \left( \sum \gamma_i U v_i \right) \]
\[ = (\gamma_1, \ldots, \gamma_n) D \begin{pmatrix} \gamma_1 \\ \vdots \\ \gamma_n \end{pmatrix} \]
\[ = \sum \lambda_i \gamma_i^2 \]

\( \lambda_i \) are eigenvalues of \( A \).
Quadratic Forms

The equation for a level set of $f$ is

$$\mathbb{R}^n : f(\mathbf{x}) = C = \left\{ \mathbf{y} \in \mathbb{R}^n : \sum_{i=1}^{n} \lambda_i y_i^2 = C \right\}$$

- If $\lambda_i \geq 0$ for all $i$, the level set is an ellipsoid, with principal axes in the directions $v_1, \ldots, v_n$. The length of the principal axis along $v_i$ is $\sqrt{C/\lambda_i}$ if $C \geq 0$ (if $\lambda_i = 0$, the level set is a degenerate ellipsoid with principal axis of infinite length in that direction). The level set is empty if $C < 0$.
  $$\Rightarrow \text{f has global min at 0, } f(x) \geq 0 \quad \forall x$$

- If $\lambda_i \leq 0$ for all $i$, the level set is an ellipsoid, with principal axes in the directions $v_1, \ldots, v_n$. The length of the principal
  $$\Rightarrow \text{f has global max at 0, } f(x) \leq 0 \quad \forall x$$
axis along $v_i$ is $\sqrt{C/\lambda_i}$ if $C \leq 0$ (if $\lambda_i = 0$, the level set is a degenerate ellipsoid with principal axis of infinite length in that direction). The level set is empty if $C > 0$.

- **If $\lambda_i > 0$ for some $i$ and $\lambda_j < 0$ for some $j$,** the level set is a hyperboloid. For example, suppose $n = 2$, $\lambda_1 > 0$, $\lambda_2 < 0$. The equation is

$$C = \lambda_1 \gamma_1^2 + \lambda_2 \gamma_2^2$$

$$= (\sqrt{\lambda_1 \gamma_1} + \sqrt{|\lambda_2| \gamma_2}) (\sqrt{\lambda_1 \gamma_1} - \sqrt{|\lambda_2| \gamma_2})$$

$\implies$ if has a saddle point at $0$

min with respect to $v_i$

max with respect to $v_j$
This is a hyperbola with asymptotes

\[
0 = \sqrt{\lambda_1 \gamma_1} + \sqrt{|\lambda_2| \gamma_2} \\
\Rightarrow \gamma_1 = -\sqrt{\frac{|\lambda_2|}{\lambda_1}} \gamma_2 \\
0 = \left( \sqrt{\lambda_1 \gamma_1} - \sqrt{|\lambda_2| \gamma_2} \right) \\
\Rightarrow \gamma_1 = \sqrt{\frac{|\lambda_2|}{\lambda_1}} \gamma_2
\]
$\lambda_1 > 0, \lambda_2 > 0$

$f$ has a global min at $0$
\[ \lambda_1 > 0, \lambda_2 < 0 \]

\[ \gamma_1 = \sqrt{|\lambda_2|/\lambda_1} \gamma_2 \]

\[ \gamma_1 = -\sqrt{|\lambda_2|/\lambda_1} \gamma_2 \]

\[ \exists x \in \mathbb{R}^n : f(x) = 0 \]

\[ f \text{ has a saddle point at } 0 \]
Quadratic Forms

This proves the following corollary of Theorem 2.

**Corollary 1.** Consider the quadratic form (1). Let \( \{v_1, \ldots, v_n\} \) be an orthonormal basis of eigenvectors of \( A \) with corresponding eigenvalues \( \{\lambda_1, \ldots, \lambda_n\} \).

1. \( f \) has a global minimum at 0 if and only if \( \lambda_i \geq 0 \) for all \( i \); the level sets of \( f \) are ellipsoids with principal axes aligned with the orthonormal eigenvectors \( v_1, \ldots, v_n \).

2. \( f \) has a global maximum at 0 if and only if \( \lambda_i \leq 0 \) for all \( i \); the level sets of \( f \) are ellipsoids with principal axes aligned with the orthonormal eigenvectors \( v_1, \ldots, v_n \).
3. If $\lambda_i < 0$ for some $i$ and $\lambda_j > 0$ for some $j$, then $f$ has a saddle point at 0; the level sets of $f$ are hyperboloids with principal axes aligned with the orthonormal eigenvectors $v_1, \ldots, v_n$. 
Bounded Linear Maps

Definition 3. Suppose $X, Y$ are normed vector spaces and $T \in L(X, Y)$. We say $T$ is bounded if

$$\exists \beta \in \mathbb{R} \text{ s.t. } \|T(x)\|_Y \leq \beta \|x\|_X \quad \forall x \in X$$

Note this implies that $T$ is Lipschitz with constant $\beta$.

why not previous notion of bounded:

$$\exists \beta \in \mathbb{R} \text{ s.t. } \|T(x)\| \leq \beta \quad \forall x$$

$T(\alpha x) = \alpha T(x) \quad \forall \alpha \in \mathbb{R}$

$\Rightarrow \quad \|T(\alpha x)\| = |\alpha| \|T(x)\| \quad \forall \alpha \in \mathbb{R}$
Bounded Linear Maps

Much more is true:

**Theorem 3** (Thms. 4.1, 4.3). Let \( X \) and \( Y \) be normed vector spaces and \( T \in L(X, Y) \). Then

\[
T \text{ is continuous at some point } x_0 \in X \iff T \text{ is continuous at every } x \in X \iff T \text{ is uniformly continuous on } X \iff T \text{ is Lipschitz} \iff T \text{ is bounded}
\]

**Proof.** Suppose \( T \) is continuous at \( x_0 \). Fix \( \varepsilon > 0 \). Then there exists \( \delta > 0 \) such that

\[
\| z - x_0 \| < \delta \Rightarrow \| T(z) - T(x_0) \| < \varepsilon
\]
Now suppose $x$ is any element of $X$. If $\|y - x\| < \delta$, let $z = y - x + x_0$, so $\|z - x_0\| = \|y - x\| < \delta$.

\[
\begin{align*}
\|T(y) - T(x)\| &= \|T(y - x)\| \\
&= \|T(y - x + x_0 - x_0)\| = \|T(z - x_0)\| \\
&< \varepsilon
\end{align*}
\]

which proves that $T$ is continuous at every $x$, and uniformly continuous.

We claim that $T$ is bounded if and only if $T$ is continuous at 0. Suppose $T$ is not bounded. Then

\[
\exists \{x_n\} \text{ s.t. } \|T(x_n)\| > n\|x_n\| \quad \forall n
\]
Note that \( x_n \neq 0 \). Let \( \varepsilon = 1 \). Fix \( \delta > 0 \) and choose \( n \) such that \( \frac{1}{n} < \delta \). Let

\[
x_n' = \frac{x_n}{n\|x_n\|} = \frac{1}{n}\frac{x_n}{\|x_n\|}
\]

\[
\|x_n'\| = \|x_n\| = \frac{1}{n}\|x_n\| < \delta
\]

\[
\|T(x_n') - T(0)\| = \|T(x_n')\| = \frac{1}{n\|x_n\|}\|T(x_n)\| > \frac{n\|x_n\|}{n\|x_n\|} = 1 = \varepsilon
\]

\[
T(x_n') = \frac{x_n}{n\|x_n\|} = \frac{1}{n}\frac{x_n}{\|x_n\|} = \frac{1}{n}\|x_n\|\|x_n\|\|x_n\| > \frac{n\|x_n\|}{n\|x_n\|} = 1 = \varepsilon
\]
Since this is true for every \( \delta \), \( T \) is not continuous at 0. Therefore, \( T \) continuous at 0 implies \( T \) is bounded. Now, suppose \( T \) is bounded, so find \( M \) such that \( \|T(x)\| \leq M\|x\| \) for every \( x \in X \). Given \( \varepsilon > 0 \), let \( \delta = \varepsilon/M \). Then

\[
\|x - 0\| < \delta \quad \Rightarrow \quad \|x\| < \delta \\
\Rightarrow \quad \|T(x) - T(0)\| = \|T(x)\| < M\delta \\
\Rightarrow \quad \|T(x) - T(0)\| < \varepsilon = M\delta
\]

so \( T \) is continuous at 0.

Thus, we have shown that continuity at some point \( x_0 \) implies uniform continuity, which implies continuity at every point, which implies \( T \) is continuous at 0, which implies that \( T \) is bounded, which implies that \( T \) is continuous at 0, which implies that \( T \) is...
continuous at some $x_0$, so all of the statements except possibly the Lipschitz statement are equivalent.

Suppose $T$ is bounded, with constant $M$. Then

$$
\|T(x) - T(y)\| = \|T(x - y)\| \leq M\|x - y\|
$$

so $T$ is Lipschitz with constant $M$; conversely, if $T$ is Lipschitz with constant $M$, then $T$ is bounded with constant $M$. So all the statements are equivalent. \[ \square \]
Bounded Linear Maps

Every linear map on a finite-dimensional normed vector space is bounded (and thus continuous, uniformly continuous, and Lipschitz continuous).

**Theorem 4** (Thm. 4.5). Let \( X \) and \( Y \) be normed vector spaces, with \( \dim X = n \). Every \( T \in L(X, Y) \) is bounded.

*Proof.* See de la Fuente.
Topological Isomorphism

Definition 4. A topological isomorphism between normed vector spaces $X$ and $Y$ is a linear transformation $T \in L(X, Y)$ that is invertible (one-to-one, onto), continuous, and has a continuous inverse.

Two normed vector spaces $X$ and $Y$ are topologically isomorphic if there is a topological isomorphism $T : X \to Y$. 
Suppose $X$ and $Y$ are normed vector spaces. We define

$$B(X,Y) = \{ T \in L(X,Y) : T \text{ is bounded} \}$$

$$\|T\|_{B(X,Y)} = \sup \left\{ \frac{\|T(x)\|_Y}{\|x\|_X}, x \in X, x \neq 0 \right\}$$

$$= \sup \{ \|T(x)\|_Y : \|x\|_X = 1 \}$$

$$\Rightarrow \|T(x)\|_Y \leq \|T\|_{B(X,Y)} \|x\|_X$$

by defn.

We skip the proofs of the rest of these results – read dLF.
The Space $B(X, Y)$

**Theorem 5** (Thm. 4.8). Let $X, Y$ be normed vector spaces. Then

$$
\left( B(X, Y), \| \cdot \|_{B(X,Y)} \right)
$$

is a normed vector space.
The Space $B(\mathbb{R}^n, \mathbb{R}^m)$

**Theorem 6** (Thm. 4.9). Let $T \in L(\mathbb{R}^n, \mathbb{R}^m)$ ($= B(\mathbb{R}^n, \mathbb{R}^m)$) with matrix $A = (a_{ij})$ with respect to the standard bases. Let

$$M = \max\{|a_{ij}| : 1 \leq i \leq m, 1 \leq j \leq n\}$$

Then

$$M \leq \|T\| \leq M \sqrt{mn}$$
Theorem 7 (Thm. 4.10). Let $R \in L(\mathbb{R}^m, \mathbb{R}^n)$ and $S \in L(\mathbb{R}^n, \mathbb{R}^p)$. Then

$$\|S \circ R\| \leq \|S\|\|R\|$$
Invertibility

Define $\Omega(\mathbb{R}^n) = \{T \in L(\mathbb{R}^n, \mathbb{R}^n) : T \text{ is invertible}\}$

**Theorem 8 (Thm. 4.11').** Suppose $T \in L(\mathbb{R}^n, \mathbb{R}^n)$ and $E$ is the standard basis of $\mathbb{R}^n$. Then

$T$ is invertible

$\iff \ker T = \{0\}$

$\iff \det (\text{Mat}_E(T)) \neq 0$

$\iff \det \left( \text{Mat}_{V,V}(T) \right) \neq 0$ for every basis $V$

$\iff \det \left( \text{Mat}_{V,W}(T) \right) \neq 0$ for every pair of bases $V, W$
Invertibility

**Theorem 9** (Thm. 4.12). If \( S, T \in \Omega(\mathbb{R}^n) \), then \( S \circ T \in \Omega(\mathbb{R}^n) \) and

\[
(S \circ T)^{-1} = T^{-1} \circ S^{-1}
\]
Invertibility

**Theorem 10** (Thm. 4.14). Let \( S, T \in L(\mathbb{R}^n, \mathbb{R}^n) \). If \( T \) is invertible and

\[
\|T - S\| < \frac{1}{\|T^{-1}\|}
\]

then \( S \) is invertible. In particular, \( \Omega(\mathbb{R}^n) \) is open in \( L(\mathbb{R}^n, \mathbb{R}^n) = B(\mathbb{R}^n, \mathbb{R}^n) \).

**Theorem 11** (Thm. 4.15). The function \((\cdot)^{-1} : \Omega(\mathbb{R}^n) \to \Omega(\mathbb{R}^n)\) that assigns \( T^{-1} \) to each \( T \in \Omega(\mathbb{R}^n) \) is continuous.