Econ 204 2018

Lecture 1

Outline

1. Introductions
2. About the Course and Other Administrative Details
3. Methods of Proof
4. Equivalence Relations
5. Cardinality
Introductions

Welcome

• 204

• Berkeley Economics

• UC Berkeley

• Berkeley

• California
US...
Introductions

- Chris Shannon
- Farzad Pourbabaee
- Walker Ray
About the Course

- **Schedule:** Lectures MTWThF 9:00 - 11:30 3108 Etcheverry Hall, often going over so don’t schedule anything before 12:00

**Sections:** MTWThF 1:00 - 2:30 and 2:30 - 4:00, in 648 Evans (please try to split up evenly)

**Office hours:** Chris Shannon MTWThF 11:30 - 12:30 (end of lecture + 1 hour) here or 511 Evans, also by appt.

Farzad + Walker MTWThF 4:00 - 5:00 648 Evans

- **Final Exam:** Wednesday August 15, 9:00 am - 12:00 pm, 10 Evans Hall
• **Prerequisites:** Math 53-54 at Berkeley or equivalent
  
  – 4 semesters college mathematics
  
  – linear algebra
  
  – multivariable calculus
  
  – rigorous approach - theorems stated carefully and some proofs given
  
  – stream for engineers and scientists
Course requirements:

- problems sets: 6 total
  (no late problem sets...no exceptions)
- exam
- reading/working on your own

Grade: 10% problem sets (5 highest scores out of 6), 90% final exam
Grading in First Year Economics Courses:

- median grade = B+ : solid command of material

- A and A- are very good grades, A+ for truly exceptional work

- B : ready to go on to further work...a B in 204 means you are ready to go on to 201a/b, 202a/b, 240a/b

- B- : very marginal, but we won’t make you take the class again. B- in 204 means you will have a very hard time in 201a/b. Recommend you take Math 53 and 54 this year, maybe Math 104, come back next year to retake 204 and
take 201a/b. B- is a passing grade, but you must maintain a B average

- C: not passing. Definitely not ready for 201a/b, 202a/b, 240a/b. Take Math 53-54 this year, maybe Math 104, retake 204 next year

- 204 with at least a B- (or a waiver from 204 requirement) is a strictly enforced prerequisite for enrollment in 201a/b

- F: means you didn’t take the final exam. Be sure to withdraw if you don’t or can’t take the final.
Resources:

Book: de la Fuente, *Mathematical Methods and Models for Economists*

Lecture notes: for every lecture + supplements for several topics

*Be sure to read Corrections Handout with dlF*

Seek out other references
This class is not normal...

- lectures

- expectations

- classroom stuff
Goals for 204

- reduce heterogeneity of math backgrounds for students in Econ graduate classes
- advance everyone’s math skills and knowledge
- present some particular concepts and results used in first-year economics courses 201a/b, 202a/b, 240a/b
- challenge everyone - so not everyone will understand everything
• develop basic math skills and knowledge needed to work as a professional economist and read academic economics

• develop ability to read and evaluate purported proofs...essential for reading and working in all branches of economics - theoretical, empirical, experimental

• develop ability to compose simple proofs...essential to working in all branches of economics - theoretical, empirical, experimental

• cover selected material from real analysis and linear algebra at moderate level of abstraction (considerably more advanced and abstract than Math 53 + 54)
• **not** to review Math 53 + 54. If you are weak on this material, take Math 53-54 this year, and take 204 next year.
Learning by Doing

- to learn this sort of mathematics you need to do more than just read the book and notes and listen to lectures

- active reading: work through each line, be sure you know how to get from one line to the next

- active listening: follow each step as we work through arguments in class

- working problems: the most valuable part of the class
• working in groups strongly encouraged...

• but, always try to work through all of the problems before talking to others

• everyone must write up his/her own solutions

• best test of understanding: can you explain it to others
Methods of Proof

What is a proof? The million dollar question...

Main Methods of Proof:

- deduction
- contraposition
- induction
- contradiction
We’ll examine each of these in turn.
Proof by Deduction

**Proof by Deduction:** A list of statements, the last of which is the statement to be proven. Each statement in the list is either

- an axiom: a fundamental assumption about mathematics, or part of definition of the object under study; or

- a previously established theorem; or

- follows from previous statements in the list by a valid rule of inference
Proof by Deduction

**Example:** Prove that the function $f(x) = x^2$ is continuous at $x = 5$.

Recall from one-variable calculus that $f(x) = x^2$ is continuous at $x = 5$ means

\[ \forall \varepsilon > 0 \ \exists \delta > 0 \text{ such that } |x - 5| < \delta \Rightarrow |f(x) - f(5)| < \varepsilon \]

That is, “for every $\varepsilon > 0$ there exists a $\delta > 0$ such that whenever $x$ is within $\delta$ of 5, $f(x)$ is within $\varepsilon$ of $f(5)$.”

To prove the claim, we must systematically verify that this definition is satisfied.
Proof. Let $\varepsilon > 0$ be given. Let

$$\delta = \min \left\{ 1, \frac{\varepsilon}{11} \right\} > 0$$

Where did that come from? Suppose $|x - 5| < \delta$. Since $\delta \leq 1$, $4 < x < 6$, so $9 < x + 5 < 11$ and $|x + 5| < 11$. Then

$$|f(x) - f(5)| = |x^2 - 25|$$
$$= |(x + 5)(x - 5)|$$
$$= |x + 5||x - 5|$$
$$< 11 \cdot \delta$$
$$\leq 11 \cdot \frac{\varepsilon}{11}$$
$$= \varepsilon$$

Thus, we have shown that for every $\varepsilon > 0$, there exists $\delta > 0$ such that $|x - 5| < \delta \Rightarrow |f(x) - f(5)| < \varepsilon$, so $f$ is continuous at $x = 5$. \qed
Proof by Contraposition

Recall some basics of logic.

¬P means “P is false.”

P ∧ Q means “P is true and Q is true.”

P ∨ Q means “P is true or Q is true (or possibly both).”

¬P ∧ Q means (¬P) ∧ Q; ¬P ∨ Q means (¬P) ∨ Q.

P ⇒ Q means “whenever P is satisfied, Q is also satisfied.”

Formally, P ⇒ Q is equivalent to ¬P ∨ Q.
Proof by Contraposition

The *contrapositive* of the statement $P \Rightarrow Q$ is the statement

$$\neg Q \Rightarrow \neg P.$$  

**Theorem 1.** $P \Rightarrow Q$ is true if and only if $\neg Q \Rightarrow \neg P$ is true.

*Proof.* Suppose $P \Rightarrow Q$ is true. Then either $P$ is false, or $Q$ is true (or possibly both). Therefore, either $\neg P$ is true, or $\neg Q$ is false (or possibly both), so $\neg (\neg Q) \lor (\neg P)$ is true, that is, $\neg Q \Rightarrow \neg P$ is true.

Conversely, suppose $\neg Q \Rightarrow \neg P$ is true. Then either $\neg Q$ is false, or $\neg P$ is true (or possibly both), so either $Q$ is true, or $P$ is false (or possibly both), so $\neg P \lor Q$ is true, so $P \Rightarrow Q$ is true. 

$\Box$
Proof by Induction

We illustrate with an example:

**Theorem 2.** For every \( n \in \mathbb{N}_0 = \{0, 1, 2, 3, \ldots\} \),

\[
\sum_{k=1}^{n} k = \frac{n(n + 1)}{2}
\]

i.e. \( 1 + 2 + \cdots + n = \frac{n(n+1)}{2} \).

**Proof.** **Base step** \( n = 0 \): LHS = \( \sum_{k=1}^{0} k = \) the empty sum = 0. \( \text{RHS} = \frac{0 \cdot 1}{2} = 0 \)

So the claim is true for \( n = 0 \).
**Induction step:** Suppose

\[
\sum_{k=1}^{n} k = \frac{n(n + 1)}{2} \quad \text{for some } n \geq 0
\]

We must show that

\[
\sum_{k=1}^{n+1} k = \frac{(n + 1)((n + 1) + 1)}{2}
\]
LHS = \[\sum_{k=1}^{n+1} k\]
\[= \sum_{k=1}^{n} k + (n + 1)\]
\[= \frac{n(n + 1)}{2} + (n + 1) \text{ by the Induction hypothesis}\]
\[= (n + 1) \left(\frac{n}{2} + 1\right)\]
\[= \frac{(n + 1)(n + 2)}{2}\]

RHS = \[\frac{(n + 1)((n + 1) + 1)}{2}\]
\[= \frac{(n + 1)(n + 2)}{2} = \text{LHS}\]

So by mathematical induction, \(\sum_{k=1}^{n} k = \frac{n(n+1)}{2}\) for all \(n \in \mathbb{N}_0\). \qed
Proof by Contradiction

Assume the negation of what is claimed, and work toward a contradiction.

**Theorem 3.** There is no rational number $q$ such that $q^2 = 2$.

*Proof.* Suppose $q^2 = 2$ where $q \in \mathbb{Q}$. Then we can write $q = \frac{m}{n}$ for some integers $m, n \in \mathbb{Z}$. Moreover, we can assume that $m$ and $n$ have no common factor; if they did, we could divide it out.

$$2 = q^2 = \frac{m^2}{n^2}$$

Therefore, $m^2 = 2n^2$, so $m^2$ is even.
We claim that $m$ is even. If not, then $m$ is odd, so $m = 2p + 1$ for some $p \in \mathbb{Z}$. Then

$$m^2 = (2p + 1)^2$$
$$= 4p^2 + 4p + 1$$
$$= 2(2p^2 + 2p) + 1$$

which is odd, contradiction. Therefore, $m$ is even, so $m = 2r$ for some $r \in \mathbb{Z}$.

$$4r^2 = (2r)^2$$
$$= m^2$$
$$= 2n^2$$

So $n^2$ is even, which implies (by the argument given above) that $n$ is even. Therefore, $n = 2s$ for some $s \in \mathbb{Z}$, so $m$ and $n$ have a
common factor, namely 2, contradiction. Therefore, there is no rational number $q$ such that $q^2 = 2$. \qed
Equivalence Relations

**Definition 1.** A binary relation $R$ from $X$ to $Y$ is a subset $R \subseteq X \times Y$. We write $xRy$ if $(x, y) \in R$ and “not $xRy$” if $(x, y) \not\in R$. $R \subseteq X \times X$ is a binary relation on $X$.

**Example:** Suppose $f : X \to Y$ is a function from $X$ to $Y$. The binary relation $R \subseteq X \times Y$ defined by

\[ xRy \iff f(x) = y \]

is exactly the graph of the function $f$. A function can be considered a binary relation $R$ from $X$ to $Y$ such that for each $x \in X$ there exists exactly one $y \in Y$ such that $(x, y) \in R$.

**Example:** Suppose $X = \{1, 2, 3\}$ and $R$ is the binary relation on $X$ given by $R = \{(1, 1), (2, 1), (2, 2), (3, 1), (3, 2), (3, 3)\}$. This is the binary relation “is weakly greater than,” or $\geq$.
Equivalence Relations

**Definition 2.** A binary relation $R$ on $X$ is

(i) reflexive if $\forall x \in X, xRx$

(ii) symmetric if $\forall x, y \in X, xRy \iff yRx$

(iii) transitive if $\forall x, y, z \in X, (xRy \land yRz) \Rightarrow xRz$

**Definition 3.** A binary relation $R$ on $X$ is an equivalence relation if it is reflexive, symmetric and transitive.
Equivalence Relations

Definition 4. Given an equivalence relation \( R \) on \( X \), write

\[
[x] = \{ y \in X : xRy \}
\]

\([x]\) is called the equivalence class containing \( x \).

The set of equivalence classes is the quotient of \( X \) with respect to \( R \), denoted \( X/R \).

Example: The binary relation \( \geq \) on \( \mathbb{R} \) is not an equivalence relation because it is not symmetric.

Example: Let \( X = \{a, b, c, d\} \) and

\[
R = \{(a, a), (a, b), (b, a), (b, b), (c, c), (c, d), (d, c), (d, d)\}
\]

\( R \) is an equivalence relation (why?) and the equivalence classes of \( R \) are \( \{a, b\} \) and \( \{c, d\} \). \( X/R = \{\{a, b\}, \{c, d\}\} \)
Equivalence Relations

The equivalence classes of an equivalence relation form a partition of $X$: every element of $X$ belongs to exactly one equivalence class.

**Theorem 4.** Let $R$ be an equivalence relation on $X$. Then $\forall x \in X, x \in [x]$. Given $x, y \in X$, either $[x] = [y]$ or $[x] \cap [y] = \emptyset$.

**Proof.** If $x \in X$, then $xRx$ because $R$ is reflexive, so $x \in [x]$.

Suppose $x, y \in X$. If $[x] \cap [y] = \emptyset$, we’re done. So suppose $[x] \cap [y] \neq \emptyset$. We must show that $[x] = [y]$, i.e. that the elements of $[x]$ are exactly the same as the elements of $[y]$.
Choose $z \in [x] \cap [y]$. Then $z \in [x]$, so $xRz$. By symmetry, $zRx$. Also $z \in [y]$, so $yRz$. By symmetry again, $zRy$. Now choose $w \in [x]$. By definition, $xRw$. Since $zRx$ and $R$ is transitive, $zRw$. By symmetry, $wRz$. Since $zRy$, $wRy$ by transitivity again. By symmetry, $yRw$, so $w \in [y]$, which shows that $[x] \subseteq [y]$. Similarly, $[y] \subseteq [x]$, so $[x] = [y]$. \qed
Cardinality

Definition 5. Two sets $A, B$ are numerically equivalent (or have the same cardinality) if there is a bijection $f : A \rightarrow B$, that is, a function $f : A \rightarrow B$ that is 1-1 ($a \neq a' \Rightarrow f(a) \neq f(a')$), and onto ($\forall b \in B \ \exists a \in A \ s.t. \ f(a) = b$).

Example: $A = \{2, 4, 6, \ldots, 50\}$ is numerically equivalent to the set $\{1, 2, \ldots, 25\}$ under the function $f(n) = 2n$.

$B = \{1, 4, 9, 16, 25, 36, 49 \ldots\} = \{n^2 : n \in \mathbb{N}\}$ is numerically equivalent to $\mathbb{N}$. 
A set is either finite or infinite. A set is \textit{finite} if it is numerically equivalent to \{1, \ldots, n\} for some \( n \). A set that is not finite is \textit{infinite}.

In particular, \( A = \{2, 4, 6, \ldots, 50\} \) is finite, \( B = \{1, 4, 9, 16, 25, 36, 49 \ldots\} \) is infinite.

A set is \textit{countable} if it is numerically equivalent to the set of natural numbers \( \mathbb{N} = \{1, 2, 3, \ldots\} \). An infinite set that is not countable is called \textit{uncountable}. 
Cardinality

**Example:** The set of integers $\mathbb{Z}$ is countable.

$$\mathbb{Z} = \{0, 1, -1, 2, -2, \ldots\}$$

Define $f : \mathbb{N} \rightarrow \mathbb{Z}$ by

\[
\begin{align*}
  f(1) &= 0 \\
  f(2) &= 1 \\
  f(3) &= -1 \\
  \vdots \\
  f(n) &= (-1)^n \left\lfloor \frac{n}{2} \right\rfloor
\end{align*}
\]

where $\lfloor x \rfloor$ is the greatest integer less than or equal to $x$. It is straightforward to verify that $f$ is one-to-one and onto.
Cardinality

Theorem 5. The set of rational numbers $\mathbb{Q}$ is countable.

“Picture Proof”:

$$\mathbb{Q} = \left\{ \frac{m}{n} : m, n \in \mathbb{Z}, n \neq 0 \right\}$$

$$= \left\{ \frac{m}{n} : m \in \mathbb{Z}, n \in \mathbb{N} \right\}$$
Go back and forth on upward-sloping diagonals, omitting the
repeats:

\[
\begin{align*}
  f(1) & = 0 \\
  f(2) & = 1 \\
  f(3) & = \frac{1}{2} \\
  f(4) & = -1 \\
  & \vdots \\
\end{align*}
\]

\[f : \mathbb{N} \to \mathbb{Q}, f \text{ is one-to-one and onto.}\]