Econ 204 2018

Lecture 6

Outline

1. Open Covers
2. Compactness
3. Sequential Compactness
4. Totally Bounded Sets
5. Heine-Borel Theorem
6. Extreme Value Theorem

Announcements:

- typo in #3 PS2
  → correction posted
- PS 2 due Tues in lecture
Open Covers

**Definition 1.** A *collection of sets*

\[ U = \{ U_\lambda : \lambda \in \Lambda \} \]

in a metric space \((X, d)\) is an open cover of \(A\) if \(U_\lambda\) is open for all \(\lambda \in \Lambda\) and

\[ \bigcup_{\lambda \in \Lambda} U_\lambda \supseteq A \]

Notice that \(\Lambda\) may be finite, countably infinite, or uncountable.
Compactness

Definition 2. A set $A$ in a metric space is compact if every open cover of $A$ contains a finite subcover of $A$. In other words, if \( \{U_\lambda : \lambda \in \Lambda\} \) is an open cover of $A$, there exist $n \in \mathbb{N}$ and $\lambda_1, \cdots, \lambda_n \in \Lambda$ such that

\[
A \subseteq U_{\lambda_1} \cup \cdots \cup U_{\lambda_n}
\]

This definition does not say “$A$ has a finite open cover” (fortunately, since this is vacuous...).

Instead for any arbitrary open cover you must specify a finite subcover of this given open cover.
Example: $(0, 1]$ is not compact in $\mathbb{E}^1$. ( $\mathbb{R}$ with standard metric)

To see this, let

$$\mathcal{U} = \left\{ U_m = \left(\frac{1}{m}, 2\right) : m \in \mathbb{N} \right\}$$

$U_m$ open U m

Then

$$\bigcup_{m \in \mathbb{N}} U_m = (0, 2) \supset (0, 1]$$

$\Rightarrow$ $\mathcal{U}$ is an open cover of $(0, 1]$
$A = (0, 1)$

$U_1 = (1, 2)$

$U_2 = (1/2, 2)$

$U_3 = (1/3, 2)$

$U_4 = (1/4, 2)$

$U_m = (\frac{1}{m}, 2)$
Given any finite subset $\{U_{m_1}, \ldots, U_{m_n}\}$ of $\mathcal{U}$, let

$$m = \max\{m_1, \ldots, m_n\}$$

Then

$$\bigcup_{i=1}^{n} U_{m_i} = U_m = \left(\frac{1}{m}, 2\right) \not\subseteq (0, 1]$$

So $(0, 1]$ is not compact.

What about $[0, 1]$? This argument doesn’t work...
Compactness

**Example:** \([0, \infty)\) is closed but not compact. (in \(\mathbb{R}\) with standard metric)

To see that \([0, \infty)\) is not compact, let

\[ U = \{ U_m = (-1, m) : m \in \mathbb{N} \} \]

Given any finite subset \(\{U_{m_1}, \ldots, U_{m_n}\}\) of \(U\), let

\[ m = \max\{m_1, \ldots, m_n\} < \infty \]

Then

\[ U_{m_1} \cup \cdots \cup U_{m_n} = (-1, m) \not\supset [0, \infty) \]
A = [0, \infty)

U_1 = (-1, 1)
U_2 = (-1, 2)
U_3 = (-1, 3)
U_m = (-1, m)

\[ A = \bigcup_{0}^{\infty} \]
A standard metric

A = \{2a, \ldots, a\} finite

A compact:

Let \( U = \bigcup_{\Lambda} U_{\lambda} \) be an open cover of A.

So, \( U_{\lambda} \) open \( \forall \lambda \in \Lambda \) and

\[ \{a, \ldots, a\} = A \subseteq U \cup_{\Lambda} \]

\( \Rightarrow \) \( A = \{a, \ldots, a\} \) still open s.t.

\( a \in U_{\lambda} \)

\( \Rightarrow \) \( \{a, \ldots, a\} \subseteq A \subseteq U_{\lambda_1} \cup U_{\lambda_2} \cup \ldots \cup U_{\lambda_n} \)
Compactness

**Theorem 1** (Thm. 8.14). *Every closed subset $A$ of a compact metric space $(X, d)$ is compact.*

**Proof.** Let $\{U_\lambda : \lambda \in \Lambda\}$ be an open cover of $A$. In order to use the compactness of $X$, we need to produce an open cover of $X$. There are two ways to do this:

$$U'_\lambda = U_\lambda \cup (X \setminus A)$$  \hspace{1cm} \text{open since $A$ closed}

$$\Lambda' = \Lambda \cup \{\lambda_0\}, \quad U_{\lambda_0} = X \setminus A$$

We choose the first path, and let

$$U'_\lambda = U_\lambda \cup (X \setminus A) \quad \forall \lambda \in \Lambda$$
\[ U_\lambda' = U_\lambda \cup (X \setminus A) \]
Since $A$ is closed, $X \setminus A$ is open; since $U_\lambda$ is open, so is $U'_\lambda$.

Then $x \in X \Rightarrow x \in A$ or $x \in X \setminus A$. If $x \in A$, $\exists \lambda \in \Lambda$ s.t. $x \in U_\lambda \subseteq U'_\lambda$. If instead $x \in X \setminus A$, then $\forall \lambda \in \Lambda, x \in U'_\lambda$. Therefore, $X \subseteq \bigcup_{\lambda \in \Lambda} U'_\lambda$, so $\{U'_\lambda : \lambda \in \Lambda\}$ is an open cover of $X$.

Since $X$ is compact,

$$\exists \lambda_1, \ldots, \lambda_n \in \Lambda \text{ s.t. } X \subseteq U'_\lambda_1 \cup \cdots \cup U'_\lambda_n$$

Then

$$a \in A \Rightarrow a \in X$$

$$\Rightarrow a \in U'_\lambda_i \text{ for some } i$$

$$\Rightarrow a \in U_\lambda_i \cup (X \setminus A)$$

$$\Rightarrow a \in U_\lambda_i \quad (\forall \in \Lambda)$$
so

\[ A \subseteq U_{\lambda_1} \cup \cdots \cup U_{\lambda_n} \]

Thus \( A \) is compact. \qed
Closed $\nRightarrow$ compact, but the converse is true: in any metric space

**Theorem 2** (Thm. 8.15). *If $A$ is a compact subset of the metric space $(X,d)$, then $A$ is closed.*

**Proof.** Suppose by way of contradiction that $A$ is not closed. Then $X \setminus A$ is not open, so we can find a point $x \in X \setminus A$ such that, for every $\varepsilon > 0$, $A \cap B_{\varepsilon}(x) \neq \emptyset$, and hence $A \cap B_{\varepsilon}[x] \neq \emptyset$. For $n \in \mathbb{N}$, let

$$U_n = X \setminus B_{\frac{1}{n}}[x]$$

open
\[ \forall x \in A \cup B_{1/n}[x] \neq \emptyset \]

\[ U_n = X \setminus B_{1/n}[x] \]

\[ a \in A \]
Each $U_n$ is open, and

$$\cup_{n\in\mathbb{N}} U_n = X \setminus \{x\} \supset A$$

since $x \notin A$. Therefore, $\{U_n : n \in \mathbb{N}\}$ is an open cover for $A$. Since $A$ is compact, there is a finite subcover $\{U_{n_1}, \ldots, U_{n_k}\}$. Let $n = \max\{n_1, \ldots, n_k\}$. Then

$$U_n = X \setminus B_{\frac{1}{n}}[x] \supseteq \cup_{j} U_{n_j} \supseteq X \setminus B_{\frac{1}{n_j}}[x] \ (j = 1, \ldots, k)$$

$$\Rightarrow \quad U_n \supseteq \cup_{j=1}^{k} U_{n_j} \supseteq A$$

But $A \cap B_{\frac{1}{n}}[x] \neq \emptyset$, so $A \nsubseteq X \setminus B_{\frac{1}{n}}[x] = U_n$, a contradiction which proves that $A$ is closed. \qed
Sequential Compactness

**Definition 3.** A set $A$ in a metric space $(X, d)$ is sequentially compact if every sequence of elements of $A$ contains a convergent subsequence whose limit lies in $A$. 
Sequential Compactness

**Theorem 3** (Thms. 8.5, 8.11). A set $A$ in a metric space $(X, d)$ is compact if and only if it is sequentially compact.

*Proof.* Suppose $A$ is compact. We will show that $A$ is sequentially compact.

If not, we can find a sequence $\{x_n\}$ of elements of $A$ such that no subsequence converges to any element of $A$. Recall that $a$ is a cluster point of the sequence $\{x_n\}$ means that

$$\forall \varepsilon > 0 \ {n : x_n \in B_\varepsilon(a)} \text{ is infinite}$$

and this is equivalent to the statement that there is a subsequence $\{x_{n_k}\}$ converging to $a$. Thus, no element $a \in A$ can be a cluster point for $\{x_n\}$, and hence

$$\forall a \in A \ \exists \varepsilon_a > 0 \ s.t. \ {n : x_n \in B_{\varepsilon_a}(a)} \text{ is finite} \quad (1)$$
Then

\{B_{\varepsilon a}(a) : a \in A\}

is an open cover of \(A\) (if \(A\) is uncountable, it will be an uncountable open cover). Since \(A\) is compact, there is a finite subcover

\(\{B_{\varepsilon a_1}(a_1), \ldots, B_{\varepsilon a_m}(a_m)\}\)

Then

\(\forall \lambda_n \subseteq A \Rightarrow\)

\(N = \{n : x_n \in (A)\} \subseteq\)

\(\subseteq \{n : x_n \in (B_{\varepsilon a_1}(a_1) \cup \cdots \cup B_{\varepsilon a_m}(a_m))\}\)

\(= \{n : x_n \in B_{\varepsilon a_1}(a_1)\} \cup \cdots \cup \{n : x_n \in B_{\varepsilon a_m}(a_m)\}\)

so \(N\) is contained in a finite union of sets, each of which is finite by Equation (1). Thus, \(N\) must be finite, a contradiction which proves that \(A\) is sequentially compact.
For the converse, see de la Fuente.
Totally Bounded Sets

Definition 4. A set $A$ in a metric space $(X, d)$ is totally bounded if, for every $\varepsilon > 0$,

$$\exists x_1, \ldots, x_n \in A \text{ s.t. } A \subseteq \bigcup_{i=1}^{n} B_\varepsilon(x_i)$$

Recall: $A \subseteq X$ bounded if $\exists \beta > 0$ and $\exists x \in X$ s.t. $A \subseteq B_\beta(x)$.
 Totally Bounded Sets

Example: Take $A = [0, 1]$ with the Euclidean metric. Given $\varepsilon > 0$, let $n > \frac{1}{\varepsilon}$. Then we may take

$$x_1 = \frac{1}{n}, x_2 = \frac{2}{n}, \ldots, x_{n-1} = \frac{n - 1}{n}$$

Then $[0, 1] \subset \bigcup_{k=1}^{n-1} B_{\varepsilon}(\frac{k}{n})$. 

\[ A = [0, 1] \]
Totally Bounded Sets

**Example:** Consider $X = [0, 1]$ with the discrete metric

$$d(x, y) = \begin{cases} 
1 & \text{if } x \neq y \\
0 & \text{if } x = y
\end{cases}$$

$X$ is not totally bounded. To see this, take $\varepsilon = \frac{1}{2}$. Then for any $x$, $B_{\varepsilon}(x) = \{x\}$, so given any finite set $x_1, \ldots, x_n$, 

$$\bigcup_{i=1}^{n} B_{\varepsilon}(x_i) = \{x_1, \ldots, x_n\} \not\subseteq [0, 1]$$

However, $X$ is bounded because $X = B_2(0)$.

bounded $\not\Rightarrow$ totally bounded
Totally Bounded Sets

Note that any totally bounded set in a metric space \((X, d)\) is also bounded. To see this, let \(A \subset X\) be totally bounded. Then \(\exists x_1, \ldots, x_n \in A\) such that \(A \subset B_1(x_1) \cup \cdots \cup B_1(x_n)\). Let

\[
M = 1 + d(x_1, x_2) + \cdots + d(x_{n-1}, x_n)
\]

Then \(M < \infty\). Now fix \(a \in A\). We claim \(d(a, x_1) < M\). To see this, notice that there is some \(n_a \in \{1, \ldots, n\}\) for which \(a \in B_1(x_{n_a})\). Then

\[
d(a, x_1) \leq d(a, x_{n_a}) + \sum_{k=1}^{n-1} d(x_k, x_{k+1})
\]

\[
< 1 + \sum_{k=1}^{n-1} d(x_k, x_{k+1})
\]

\[
= M
\]
Totally Bounded Sets

Remark 4. Every compact subset of a metric space is totally bounded:

\((\varepsilon > 0)\)

Fix \(\varepsilon\) and consider the open cover

\[ U_{\varepsilon} = \{ B_{\varepsilon}(a) : a \in A \} \]

If \(A\) is compact, then every open cover of \(A\) has a finite subcover; in particular, \(U_{\varepsilon}\) must have a finite subcover, but this just says that \(A\) is totally bounded.

\[ \Rightarrow \exists \ a_1, \ldots, a_n \in A \ \text{st.} \]

\[ A \subseteq B_{\varepsilon}(a_1) \cup \cdots \cup B_{\varepsilon}(a_n) \]

Converse false: e.g. \((0, 1]\) is totally bounded but not compact.
Compactness and Totally Bounded Sets

**Theorem 5** (Thm. 8.16). Let $A$ be a subset of a metric space $(X, d)$. Then $A$ is compact if and only if it is complete and totally bounded.

**Proof.** Here is a sketch of the proof; see de la Fuente for details. Compact implies totally bounded (Remark 4). Suppose $\{x_n\}$ is a Cauchy sequence in $A$. Since $A$ is compact, $A$ is sequentially compact, hence $\{x_n\}$ has a convergent subsequence $x_{n_k} \to a \in A$. Since $\{x_n\}$ is Cauchy, $x_n \to a$ (why?), so $A$ is complete.

Conversely, suppose $A$ is complete and totally bounded. Let $\{x_n\}$ be a sequence in $A$. Because $A$ is totally bounded, we can extract a Cauchy subsequence $\{x_{n_k}\}$ (why?). Because $A$ is complete, $x_{n_k} \to a$ for some $a \in A$, which shows that $A$ is sequentially compact and hence compact. □
Compact $\iff$ Closed and Totally Bounded

Putting these together: with results from lecture 5:

**Corollary 1.** Let $A$ be a subset of a complete metric space $(X, d)$. Then $A$ is compact if and only if $A$ is closed and totally bounded.

$(X, d)$ complete, $A \subseteq X$ then:

- $A$ compact $\Rightarrow$ $A$ complete and totally bounded
- $A$ closed and totally bounded $\Rightarrow$ $A$ compact
Example: $[0, 1]$ is compact in $E^1$. (\(\mathbb{R}\) with standard metric)

$E^1$ is complete, $[0, 1]$ is closed and totally bounded

$\Rightarrow [0, 1]$ is compact

Note: compact \(\Rightarrow\) closed and bounded, but converse need not be true.

E.g. $[0, 1]$ with the discrete metric.

$[0, 1]$ with discrete metric is closed and bounded

but not totally bounded, so not compact
Heine-Borel Theorem - $E^1$

**Theorem 6** (Thm. 8.19, Heine-Borel). If $A \subseteq E^1$, then $A$ is compact if and only if $A$ is closed and bounded.

\[\Leftarrow\] **Proof.** Let $A$ be a closed, bounded subset of $\mathbb{R}$. Then $A \subseteq [a,b]$ for some interval $[a,b]$. Let \( \{x_n\} \) be a sequence of elements of $[a,b]$. By the Bolzano-Weierstrass Theorem, \( \{x_n\} \) contains a convergent subsequence with limit $x \in \mathbb{R}$. Since $[a,b]$ is closed, $x \in [a,b]$. Thus, we have shown that $[a,b]$ is sequentially compact, hence compact. $A$ is a closed subset of $[a,b]$, hence $A$ is compact.

Conversely, if $A$ is compact, $A$ is closed and bounded.
Heine-Borel Theorem - $\mathbb{E}^n$

**Theorem 7** (Thm. 8.20, Heine-Borel). If $A \subseteq \mathbb{E}^n$, then $A$ is compact if and only if $A$ is closed and bounded.

Proof. See de la Fuente. □

**Example:** The closed interval

$$[a, b] = \{x \in \mathbb{R}^n : a_i \leq x_i \leq b_i \text{ for each } i = 1, \ldots, n\}$$

is compact in $\mathbb{E}^n$ for any $a, b \in \mathbb{R}^n$.\[23\]
Continuous Images of Compact Sets

**Theorem 8** (8.21). Let \((X, d)\) and \((Y, \rho)\) be metric spaces. If \(f : X \to Y\) is continuous and \(C\) is a compact subset of \((X, d)\), then \(f(C)\) is compact in \((Y, \rho)\).

**Proof.** There is a proof in de la Fuente using sequential compactness. Here we give an alternative proof using directly the open cover definition of compactness.

Let \(\{U_\lambda : \lambda \in \Lambda\}\) be an open cover of \(f(C)\). For each point \(c \in C\), \(f(c) \in f(C)\) so \(f(c) \in U_{\lambda_c}\) for some \(\lambda_c \in \Lambda\), that is, \(c \in f^{-1}(U_{\lambda_c})\). Thus the collection \(\{f^{-1}(U_\lambda) : \lambda \in \Lambda\}\) is a cover of \(C\); in addition, since \(f\) is continuous, each set \(f^{-1}(U_\lambda)\) is
open in $C$, so $\{ f^{-1}(U_\lambda) : \lambda \in \Lambda \}$ is an open cover of $C$. Since $C$ is compact, there is a finite subcover

$$\{ f^{-1}(U_{\lambda_1}), \ldots, f^{-1}(U_{\lambda_n}) \}$$

of $C$. Given $x \in f(C)$, there exists $c \in C$ such that $f(c) = x$, and $c \in f^{-1}(U_{\lambda_i})$ for some $i$, so $x \in U_{\lambda_i}$. Thus, $\{ U_{\lambda_1}, \ldots, U_{\lambda_n} \}$ is a finite subcover of $f(C)$, so $f(C)$ is compact. \qed
Extreme Value Theorem

Corollary 2 (Thm. 8.22, Extreme Value Theorem). Let $C$ be a compact set in a metric space $(X,d)$, and suppose $f : C \to \mathbb{R}$ is continuous. Then $f$ is bounded on $C$ and attains its minimum and maximum on $C$.

Proof. $f(C)$ is compact by Theorem 8.21, hence closed and bounded. Let $M = \sup f(C); \ M < \infty$. Then $\forall m > 0$ there exists $y_m \in f(C)$ such that

$$M - \frac{1}{m} \leq y_m \leq M$$

So $y_m \to M$ and $\{y_m\} \subseteq f(C)$. Since $f(C)$ is closed, $M \in f(C)$, i.e. there exists $c \in C$ such that $f(c) = M = \sup f(C)$, so $f$ attains its maximum at $c$. The proof for the minimum is similar. □
Compactness and Uniform Continuity

**Theorem 9 (Thm. 8.24).** Let \((X, d)\) and \((Y, \rho)\) be metric spaces, \(C\) a compact subset of \(X\), and \(f : C \to Y\) continuous. Then \(f\) is uniformly continuous on \(C\).

**Proof.** Fix \(\varepsilon > 0\). We ignore \(X\) and consider \(f\) as defined on the metric space \((C, d)\). Given \(c \in C\), find \(\delta(c) > 0\) such that

\[
x \in C, \quad d(x, c) < 2\delta(c) \Rightarrow \rho(f(x), f(c)) < \frac{\varepsilon}{2}
\]

Let

\[
U_c = B_{\delta(c)}(c)
\]

Then

\[
\{U_c : c \in C\}
\]
is an open cover of $C$. Since $C$ is compact, there is a finite subcover

$$\{U_{c_1}, \ldots, U_{c_n}\}$$

Let

$$\delta = \min\{\delta(c_1), \ldots, \delta(c_n)\} > 0$$

Given $x, y \in C$ with $d(x, y) < \delta$, note that $x \in U_{c_i}$ for some $i \in \{1, \ldots, n\}$, so $d(x, c_i) < \delta(c_i)$.

$$d(y, c_i) \leq d(y, x) + d(x, c_i)$$

$$< \delta + \delta(c_i)$$

$$\leq \delta(c_i) + \delta(c_i)$$

$$= 2\delta(c_i)$$
so

\[ \rho(f(x), f(y)) \leq \frac{\rho(f(x), f(c_i)) + \rho(f(c_i), f(y))}{\varepsilon} + \frac{\varepsilon}{2} \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \]

which proves that \( f \) is uniformly continuous.