Section 3.3. Quotient Vector Spaces

Given a vector space $X$ over a field $F$ and a vector subspace $W$ of $X$, define an equivalence relation by

$$x \sim y \iff x - y \in W$$

Form a new vector space $X/W$: the set of elements of $X/W$ is

$$\{ [x] : x \in X \}$$

where $[x]$ denotes the equivalence class of $x$ with respect to $\sim$. $X/W$ is read “$X$ mod $W$”. Note that the vectors in $X/W$ are sets of vectors in $X$: for $x \in X$,

$$[x] = \{ x + w : w \in W \}$$

We claim that $X/W$ can be viewed as a vector space over $F$. Define the vector space operations $+ , \cdot$ in $X/W$ as follows:

$$[x] + [y] = [x + y]$$
$$\alpha [x] = [\alpha x]$$

The exercise below asks you to verify that these operations are well-defined. Then $X/W$ is a vector space over $F$ with these definitions for $+$ and $\cdot$.

**Exercise:** Verify that $\sim$ above is an equivalence relation and that vector addition and scalar multiplication are well-defined, i.e.

$$[x] = [x'], [y] = [y'] \Rightarrow [x + y] = [x' + y']$$
$$[x] = [x'], \alpha \in F \Rightarrow [\alpha x] = [\alpha x']$$

**Example:** Let $X = \mathbb{R}^3$ and let $W = \{ x \in \mathbb{R}^3 : x_1 = x_2 = 0 \}$. Then for $x, y \in \mathbb{R}^3$,

$$x \sim y \iff x - y \in W$$
$$\iff x_1 - y_1 = 0, x_2 - y_2 = 0$$
$$\iff x_1 = y_1, x_2 = y_2$$

and

$$[x] = \{ x + w : w \in W \} = \{ (x_1, x_2, z) : z \in \mathbb{R} \}$$

So the equivalence class corresponding to $x$ is the line in $\mathbb{R}^3$ through $x$ parallel to the axis of the third coordinate. See Figure 1. What is $X/W$? Intuitively this equivalence relation ignores the third coordinate, and we can identify the equivalence class $[x]$ with the vector $(x_1, x_2) \in \mathbb{R}^2$. The next two results show how to formalize this connection.

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1The first part of this material is not in de la Fuente.
Theorem 1 If $X$ is a vector space with $\dim X = n$ for some $n \in \mathbb{N}$ and $W$ is a vector subspace of $X$, then

$$\dim(X/W) = \dim X - \dim W$$

Proof: (Sketch) Begin with a basis $\{w_1, \ldots, w_c\}$ for $W$, and a basis $\{[x_1], \ldots, [x_k]\}$ for $X/W$. Show that

$$\{w_1, \ldots, w_c\} \cup \{x_1, \ldots, x_k\}$$

is a basis for $X$. ■

Theorem 2 Let $X$ and $Y$ be vector spaces over the same field $F$ and $T \in L(X, Y)$. Then $\text{Im } T$ is isomorphic to $X/\ker T$.

Proof: Notice that if $X$ is finite-dimensional, then

$$\dim(X/\ker T) = \dim X - \dim \ker T \quad \text{(by the previous theorem)}$$

$$= \text{Rank } T \quad \text{(by the Rank-Nullity Theorem)}$$

$$= \dim \text{Im } T$$

so $X/\ker T$ is isomorphic to $\text{Im } T$.

We prove that this is true in general, and that the isomorphism is natural.

Define $\tilde{T} : X/\ker T \to \text{Im } T$ by

$$\tilde{T}([x]) = T(x)$$

We first need to check that this is well-defined, that is, that if $[x] = [x']$ then $\tilde{T}([x]) = \tilde{T}([x'])$.

$$[x] = [x'] \Rightarrow x \sim x'$$

$$\Rightarrow x - x' \in \ker T$$

$$\Rightarrow T(x - x') = 0$$

$$\Rightarrow T(x) = T(x')$$

so $\tilde{T}$ is well-defined.

Clearly, $\tilde{T} : X/\ker T \to \text{Im } T$. It is easy to check that $\tilde{T}$ is linear, so $\tilde{T} \in L(X/\ker T, \text{Im } T)$. Next we show that $\tilde{T}$ is an isomorphism.

$$\tilde{T}([x]) = \tilde{T}([y]) \Rightarrow T(x) = T(y)$$

$$\Rightarrow T(x - y) = 0$$

$$\Rightarrow x - y \in \ker T$$

$$\Rightarrow x \sim y$$

$$\Rightarrow [x] = [y]$$
so $\tilde{T}$ is one-to-one.

\[
y \in \text{Im} T \Rightarrow \exists x \in X \text{ s.t. } T(x) = y
\]

\[
\Rightarrow \tilde{T}([x]) = y
\]

so $\tilde{T}$ is onto, hence $\tilde{T}$ is an isomorphism. ■

**Example:** Consider $T \in L(\mathbb{R}^3, \mathbb{R}^2)$ defined by

\[
T(x_1, x_2, x_3) = (x_1, x_2)
\]

Then ker$T = \{x \in \mathbb{R}^3 : x_1 = x_2 = 0\}$ is the $x_3$-axis. (Also notice ker$T = W$ from the previous example.)

Given $x$, the equivalence class $[(x_1, x_2, x_3)]$ is just the line through $x$ parallel to the $x_3$-axis. $\tilde{T}([x]) = T(x_1, x_2, x_3) = (x_1, x_2)$.

\[
\text{Im} T = \mathbb{R}^2, \quad X/\ker T \cong \mathbb{R}^2 = \text{Im} T
\]

as we suggested intuitively above (here the symbol $\cong$ denotes isomorphism, that is, we write $Y \cong Z$ if $Y$ and $Z$ are isomorphic.)

Every real vector space $X$ with dimension $n$ is isomorphic to $\mathbb{R}^n$. What’s the isomorphism?

Let $X$ be a finite-dimensional vector space over $\mathbb{R}$ with dim$X = n$. Fix any Hamel basis $V = \{v_1, \ldots, v_n\}$ of $X$. Any $x \in X$ has a unique representation

\[
x = \sum_{j=1}^n \beta_j v_j
\]

(here, we allow $\beta_j = 0$). (Generally, vectors are represented as column vectors, not row vectors.) Then given the representation of $x$ above, we write

\[
crd_V(x) = \begin{pmatrix}
\beta_1 \\
\vdots \\
\beta_n
\end{pmatrix} \in \mathbb{R}^n
\]

That is, $crd_V(x)$ is the vector of coordinates of $x$ with respect to the basis $V$.

\[
crd_V(v_1) = \begin{pmatrix}
1 \\
0 \\
0
\end{pmatrix}, \quad crd_V(v_2) = \begin{pmatrix}
0 \\
1 \\
0
\end{pmatrix}, \quad crd_V(v_n) = \begin{pmatrix}
0 \\
0 \\
1
\end{pmatrix}
\]

$crd_V$ is an isomorphism from $X$ to $\mathbb{R}^n$. 3
Matrix Representation of a Linear Transformation

Suppose $T \in L(X,Y)$, $\dim X = n$ and $\dim Y = m$. Fix bases

\[ V = \{v_1, \ldots, v_n\} \text{ of } X \]
\[ W = \{w_1, \ldots, w_m\} \text{ of } Y \]

$T(v_j) \in Y$, so

\[ T(v_j) = \sum_{i=1}^{m} \alpha_{ij} w_i \]

Define

\[
M_{x_{W,Y}}(T) = \begin{pmatrix}
\alpha_{11} & \cdots & \alpha_{1n} \\
\vdots & \ddots & \vdots \\
\alpha_{m1} & \cdots & \alpha_{mn}
\end{pmatrix}
\]

Notice that the columns are the coordinates (expressed with respect to $W$) of $T(v_1), \ldots, T(v_n)$.

Observe

\[
\begin{pmatrix}
\alpha_{11} & \cdots & \alpha_{1n} \\
\vdots & \ddots & \vdots \\
\alpha_{m1} & \cdots & \alpha_{mn}
\end{pmatrix}
\begin{pmatrix}
1 \\
0 \\
0
\end{pmatrix}
= 
\begin{pmatrix}
\alpha_{11} \\
\vdots \\
\alpha_{m1}
\end{pmatrix}
\]

so

\[
M_{x_{W,Y}}(T) \cdot \text{crd}_V(v_j) = \text{crd}_W(T(v_j))
\]
\[
M_{x_{W,Y}}(T) \cdot \text{crd}_V(x) = \text{crd}_W(T(x)) \quad \forall x \in X
\]

Multiplying a vector by a matrix does two things:

- Computes the action of $T$
- Accounts for the change in basis

**Example**: $X = Y = \mathbb{R}^2$, $V = \{(1,0), (0,1)\}$, $W = \{(1,1), (-1,1)\}$, $T = id$, that is, $T(x) = x$ for all $x$.

\[
M_{x_{W,Y}}(T) \neq \begin{pmatrix}
1 & 0 \\
0 & 1
\end{pmatrix}
\]

$M_{x_{W,Y}}(T)$ is the matrix that changes basis from $V$ to $W$. How do we compute it?

\[
v_1 = (1,0) = \alpha_{11}(1,1) + \alpha_{21}(-1,1)
\]
\[
\alpha_{11} - \alpha_{21} = 1
\]
\[
\alpha_{11} + \alpha_{21} = 0
\]
\[
2\alpha_{11} = 1, \alpha_{11} = \frac{1}{2}
\]
\[
\begin{align*}
\alpha_{21} & = -\frac{1}{2} \\
v_2 &= (0, 1) = \alpha_{12}(1, 1) + \alpha_{22}(-1, 1) \\
\alpha_{12} - \alpha_{22} & = 0 \\
\alpha_{12} + \alpha_{22} & = 1 \\
2\alpha_{12} & = 1, \alpha_{12} = \frac{1}{2} \\
\alpha_{22} & = \frac{1}{2}
\end{align*}
\]

\[Mtx_{W,V}(id) = \begin{pmatrix}
\frac{1}{2} & \frac{1}{2} \\
-\frac{1}{2} & \frac{1}{2}
\end{pmatrix}\]

**Theorem 3 (Thm. 3.5')** Let \(X\) and \(Y\) be vector spaces over the same field \(F\), with \(\dim X = n\), \(\dim Y = m\). Then \(L(X,Y)\), the space of linear transformations from \(X\) to \(Y\), is isomorphic to \(F_{m \times n}\), the vector space of \(m \times n\) matrices over \(F\). If \(V = \{v_1, \ldots, v_n\}\) is a basis for \(X\) and \(W = \{w_1, \ldots, w_m\}\) is a basis for \(Y\), then

\[Mtx_{W,V} \in L(L(X,Y), F_{m \times n})\]

and \(Mtx_{W,V}\) is an isomorphism from \(L(X,Y)\) to \(F_{m \times n}\).

**Theorem 4 (From Handout)** Let \(X, Y, Z\) be finite-dimensional vector spaces over the same field \(F\) with bases \(U, V, W\) respectively. Let \(S \in L(X,Y)\) and \(T \in L(Y,Z)\). Then

\[Mtx_{W,V}(T) \cdot Mtx_{V,U}(S) = Mtx_{W,U}(T \circ S)\]

i.e. matrix multiplication corresponds via the matrix representation isomorphism to composition of linear transformations.

**Proof:** See handout. \(\blacksquare\)

Note that \(Mtx_{W,V}\) is a function from \(L(X,Y)\) to the space \(F_{m \times n}\) of \(m \times n\) matrices, while \(Mtx_{W,V}(T)\) is an \(m \times n\) matrix.

The theorem can be summarized by the following “Commutative Diagram:"

\[
\begin{array}{ccc}
S & \xrightarrow{T} & T \\
\text{crd}_U \downarrow & & \downarrow \text{crd}_V \\
\mathbb{R}^n & \rightarrow & \mathbb{R}^m & \rightarrow & \mathbb{R}^r \\
Mtx_{V,U}(S) & \rightarrow & Mtx_{W,V}(T)
\end{array}
\]

We say the diagram commutes because you get the same answer any way you go around the diagram (in directions allowed by the arrows). The \(\text{crd}\) arrows go in both directions because \(\text{crd}\) is an isomorphism.
Section 3.5. Change of Basis and Similarity

Let $X$ be a finite-dimensional vector space with basis $V$. If $T \in L(X, X)$ it is customary to use the same basis in the domain and range. In this case, $Mtx_V(T)$ denotes $Mtx_{V,V}(T)$

**Question:** If $W$ is another basis for $X$, how are $Mtx_V(T)$ and $Mtx_W(T)$ related?

$$Mtx_{V,W}(id) \cdot Mtx_W(T) \cdot Mtx_{W,V}(id) = Mtx_{V,W}(id) \cdot Mtx_{W,V}(T \circ id) = Mtx_{V,V}(id \circ T \circ id) = Mtx_V(T)$$

and

$$Mtx_{V,W}(id) \cdot Mtx_{W,V}(id) = Mtx_{V,V}(id)$$

$$= \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 1 \end{pmatrix}$$

So this says that

$$Mtx_V(T) = P^{-1}Mtx_W(T)P$$

for the invertible matrix

$$P = Mtx_{W,V}(id)$$

that is the change of basis matrix. On the other hand, if $P$ is any invertible matrix, then $P$ is also a change of basis matrix for appropriate corresponding bases (see handout).

**Definition 5** Square matrices $A$ and $B$ are similar if

$$A = P^{-1}BP$$

for some invertible matrix $P$.

**Theorem 6** Suppose that $X$ is a finite-dimensional vector space.

1. If $T \in L(X, X)$ then any two matrix representations of $T$ are similar. That is, if $U, W$ are any two bases of $X$, then $Mtx_W(T)$ and $Mtx_U(T)$ are similar.

2. Conversely, two similar matrices represent the same linear transformation $T$, relative to suitable bases. That is, given similar matrices $A, B$ with $A = P^{-1}BP$ and any basis $U$, there is a basis $W$ and $T \in L(X, X)$ such that

$$B = Mtx_U(T)$$

$$A = Mtx_W(T)$$

$$P = Mtx_{U,W}(id)$$

$$P^{-1} = Mtx_{W,U}(id)$$
Proof: See Handout on Diagonalization and Quadratic Forms. ■

Section 3.6. Eigenvalues and Eigenvectors

Here, we define eigenvalues and eigenvectors of a linear transformation and show that \( \lambda \) is an eigenvalue of \( T \) if and only if \( \lambda \) is an eigenvalue for some matrix representation of \( T \) if and only if \( \lambda \) is an eigenvalue for every matrix representation of \( T \).

Definition 7 Let \( X \) be a vector space and \( T \in L(X, X) \). We say that \( \lambda \) is an eigenvalue of \( T \) and \( v \neq 0 \) is an eigenvector corresponding to \( \lambda \) if \( T(v) = \lambda v \).

Theorem 8 (Theorem 4 in Handout) Let \( X \) be a finite-dimensional vector space, and \( U \) a basis. Then \( \lambda \) is an eigenvalue of \( T \) if and only if \( \lambda \) is an eigenvalue of \( \text{Mtx}_U(T) \). \( v \) is an eigenvector of \( T \) corresponding to \( \lambda \) if and only if \( \text{crd}_U(v) \) is an eigenvector of \( \text{Mtx}_U(T) \) corresponding to \( \lambda \).

Proof: By the Commutative Diagram Theorem,

\[
T(v) = \lambda v \iff \text{crd}_U(T(v)) = \text{crd}_U(\lambda v) \\
\iff \text{Mtx}_U(T)(\text{crd}_U(v)) = \lambda(\text{crd}_U(v))
\]

Computing eigenvalues and eigenvectors:

Suppose \( \text{dim } X = n \); let \( I \) be the \( n \times n \) identity matrix. Given \( T \in L(X, X) \), fix a basis \( U \) and let

\[
A = \text{Mtx}_U(T)
\]

Find the eigenvalues of \( T \) by computing the eigenvalues of \( A \):

\[
Av = \lambda v \iff (A - \lambda I)v = 0 \\
\iff (A - \lambda I) \text{ is not invertible} \\
\iff \text{det}(A - \lambda I) = 0
\]
We have the following facts:

- If \( A \in \mathbb{R}^{n \times n} \),
  \[
  f(\lambda) = \det(A - \lambda I)
  \]
  is an \( n \)th degree polynomial in \( \lambda \) with real coefficients; it is called the characteristic polynomial of \( A \).
- \( f \) has \( n \) roots in \( \mathbb{C} \), counting multiplicity:
  \[
  f(\lambda) = (\lambda - c_1)(\lambda - c_2) \cdots (\lambda - c_n)
  \]
  where \( c_1, \ldots, c_n \in \mathbb{C} \) are the eigenvalues; the \( c_j \)'s are not necessarily distinct. Notice that \( f(\lambda) = 0 \) if and only if \( \lambda \in \{c_1, \ldots, c_n\} \), so the roots are the solutions of the equation \( f(\lambda) = 0 \).
- the roots that are not real come in conjugate pairs:
  \[
  f(a + bi) = 0 \iff f(a - bi) = 0
  \]
- if \( \lambda = c_j \in \mathbb{R} \), there is a corresponding eigenvector in \( \mathbb{R}^n \).
- if \( \lambda = c_j \not\in \mathbb{R} \), the corresponding eigenvectors are in \( \mathbb{C}^n \setminus \mathbb{R}^n \).

**Diagonalization**

**Definition 9** Suppose \( X \) is a finite-dimensional vector space with basis \( U \). Given a linear transformation \( T \in L(X, X) \), let

\[
A = M_{xU}(T)
\]

We say that \( A \) can be diagonalized (or is diagonalizable) if there is a basis \( W \) for \( X \) such that \( M_{xW}(T) \) is diagonal, i.e.

\[
M_{xW}(T) = \begin{pmatrix}
\lambda_1 & 0 & 0 & \cdots & 0 & 0 \\
0 & \lambda_2 & 0 & \cdots & 0 & 0 \\
\vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & 0 & \lambda_n
\end{pmatrix}
\]

Notice that the eigenvectors of \( M_{xW}(T) \) are exactly the standard basis vectors of \( \mathbb{R}^n \). But \( w_j \) is an eigenvector of \( T \) corresponding to \( \lambda_j \) if and only if \( crd_W(w_j) \) is an eigenvector of \( M_{xW}(T) \), and \( crd_W(w_j) \) is the \( j \)th standard basis vector of \( \mathbb{R}^n \), so \( W = \{w_1, \ldots, w_n\} \) where \( w_j \) is an eigenvector corresponding to \( \lambda_j \).

Then the action of \( T \) is clear: it stretches each basis element \( w_i \) by the factor \( \lambda_i \).
Theorem 10 (Thm. 6.7') Let $X$ be an $n$-dimensional vector space, $T \in L(X, X)$, $U$ any basis of $X$, and $A = Mtx_U(T)$. Then the following are equivalent:

1. $A$ can be diagonalized
2. there is a basis $W$ for $X$ consisting of eigenvectors of $T$
3. there is a basis $V$ for $\mathbb{R}^n$ consisting of eigenvectors of $A$

Proof: Follows from Theorem 6.7 in de la Fuente and Theorem 4 from the Handout. ■

Theorem 11 (Thm. 6.8') Let $X$ be a vector space and $T \in L(X, X)$.

1. If $\lambda_1, \ldots, \lambda_m$ are distinct eigenvalues of $T$ with corresponding eigenvectors $v_1, \ldots, v_m$, then $\{v_1, \ldots, v_m\}$ is linearly independent.
2. If $\dim X = n$ and $T$ has $n$ distinct eigenvalues, then $X$ has a basis consisting of eigenvectors of $T$; consequently, if $U$ is any basis of $X$, then $Mtx_U(T)$ is diagonalizable.

Proof: This is an adaptation of the proof of Theorem 6.8 in de la Fuente. ■
Figure 1: An illustration of $X/W$ where $X = \mathbb{R}^3$ and $W = \{x \in \mathbb{R}^3 : x_1 = x_2 = 0\}$. Here $[x] = \{(x_1, x_2, z) : z \in \mathbb{R}\}$ is the line through $x$ parallel to the axis of the third coordinate.