Econ 204 – Problem Set 1

Due Friday July 27 1

- 1. Use induction to prove the following:
 - (a) $2^{2n} 1$ is divisible by 3 for all $n \in \mathbb{N}$.
 - (b) $1 + \frac{1}{\sqrt{2}} + \ldots + \frac{1}{\sqrt{n}} \le 2\sqrt{n}$
- 2. In the following examples, show that the sets A and B are numerically equivalent by finding a specific bijection between the two.
 - (a) A = [0, 1], B = [10, 20]
 - (b) A = [0, 1], B = [0, 1)
 - (c) $A = (-1, 1), B = \mathbb{R}$
- 3. Define the infinite **cartesian product** of a set X with itself as $X^{\omega} := \prod_{i \in \mathbb{N}} X$. Prove by contradiction that for $X = \{0, 1\}$, X^{ω} is uncountable. (Hint: suppose there exists a surjective map $f : \mathbb{N} \to X^{\omega}$, and find an element in X^{ω} which is not in the image of f).
- 4. (Dynkins's π - λ system Theorem): The goal of this exercise is to prove this theorem, and review/practice some of the set theoretical results.

Suppose Ω is some arbitrary set (which need not have any topological or algebraic structure). Then, $\mathscr{F} \subset 2^{\Omega}$ as a collection of subsets of Ω is called a σ -algebra if it satisfies following properties:

- $\emptyset \in \mathscr{F}$.
- For every $A \subset \Omega$ where $A \in \mathscr{F}$, $A^c \in \mathscr{F}$ (A^c refers to the complement of set A, i.e $\Omega \setminus A$).
- For every *countable* sequence of subsets $\{A_n\}$ where $A_n \in \mathscr{F}$ for all $n, \bigcup_n A_n \in \mathscr{F}$.
- (a) Show that $\Omega \in \mathscr{F}$.
- (b) Prove that \mathscr{F} is closed under countable intersection.

Two more definitions: first, $\Lambda \subset 2^{\Omega}$ is called a λ -system if:

- $\Omega \in \Lambda$.
- If $A, B \in \Lambda$ and $A \subset B$, then $B \setminus A \in \Lambda$.
- If $\{A_n\}$ is an increasing sequence of subsets, i.e $A_1 \subset A_2 \subset \ldots$, with each element being in Λ , then $\bigcup_n A_n \in \Lambda$.

Second, $\Pi \subset 2^{\Omega}$ is called a π -system, if it is closed under *finite* intersection. Now assume Π is a π -system such that $\Pi \subset \Lambda$, where Λ is a λ -system. The Dynkin's theorem which we want to prove states that the smallest σ -algebra containing Π (denoted by $\sigma(\Pi)$) is a subset of Λ . Try to keep on with each step below until the final result drops out:

¹Please keep your answers short and concise. The solution to each (sub)question could well fit in at most one page.

- (c) Let $\lambda(\Pi)$ be the smallest² λ -system containing Π . Explain why $\lambda(\Pi) \subset \Lambda$. (Hint: note that $\Pi \subset \Lambda$).
- (d) Let $B \in \Pi$ and define $\mathcal{A}_B := \{A \subset \Omega : A \cap B \in \lambda(\Pi)\}$. Show that \mathcal{A}_B is itself a λ -system and contains $\lambda(\Pi)$, i.e $\lambda(\Pi) \subset \mathcal{A}_B$.
- (e) Now let $A \in \lambda(\Pi)$, and define $\mathcal{B}_A := \{B \subset \Omega : A \cap B \in \lambda(\Pi)\}$. Again, show that \mathcal{B}_A is a λ -system containing $\lambda(\Pi)$.
- (f) Use the previous two steps to show $\lambda(\Pi)$ is also a π -system.
- (g) Given that we have seen so far that $\lambda(\Pi)$ is both a π -system and a λ -system, deduce that it has to be a σ -algebra.
- (h) Conclude that $\sigma(\Pi) \subset \Lambda$.
- 5. Let \mathcal{U} and \mathcal{Z} be two sets, and $(u, z) \mapsto P(u, z)$ be a bounded function on $\mathcal{U} \times \mathcal{Z}$. Define the *upper and lower value functions* as:

$$V_{+} = \inf_{u \in \mathcal{U}} \sup_{z \in \mathcal{Z}} P(u, z)$$

$$V_{-} = \sup_{z \in \mathcal{Z}} \inf_{u \in \mathcal{U}} P(u, z)$$
(1)

- (a) Show that $V_+ \geq V_-$.
- (b) Call any function $\beta : \mathcal{U} \to \mathcal{Z}$ a *strategy* for the maximizing side. Denote the space of all such strategies as \mathscr{B} . Prove the following identity, and explain why it is not in contrast with part (a).

$$V_{+} = \inf_{u \in \mathcal{U}} \sup_{\beta \in \mathscr{B}} P(u, \beta(u)) = \sup_{\beta \in \mathscr{B}} \inf_{u \in \mathcal{U}} P(u, \beta(u))$$
(2)

6. Let $\{a_n\}$ and $\{b_n\}$ be two sequences of non-negative real numbers. Further, let $\{s_n\}$ be a monotone positive sequence satisfying $s_n \to \infty$. Prove that

$$\limsup_{n \to \infty} \left(a_n^{s_n} + b_n^{s_n} \right)^{1/s_n} = \max \left\{ \limsup_{n \to \infty} a_n, \limsup_{n \to \infty} b_n \right\}.$$
 (3)

(Hint: recall that $\limsup_{n\to\infty} a_n = \inf_{m\in\mathbb{N}} \sup_{n\ge m} a_n$)

²By smallest we mean: $\lambda(\Pi) = \bigcap_{\{\Lambda_{\alpha} \text{ is } \lambda \text{-system in } 2^{\Omega}: \Pi \subset \Lambda_{\alpha}\}} \Lambda_{\alpha}$