## Econ 204 - Problem Set 1

Due Friday July $27^{1}$

1. Use induction to prove the following:
(a) $2^{2 n}-1$ is divisible by 3 for all $n \in \mathbb{N}$.
(b) $1+\frac{1}{\sqrt{2}}+\ldots+\frac{1}{\sqrt{n}} \leq 2 \sqrt{n}$
2. In the following examples, show that the sets $A$ and $B$ are numerically equivalent by finding a specific bijection between the two.
(a) $A=[0,1], B=[10,20]$
(b) $A=[0,1], B=[0,1)$
(c) $A=(-1,1), B=\mathbb{R}$
3. Define the infinite cartesian product of a set $X$ with itself as $X^{\omega}:=\prod_{i \in \mathbb{N}} X$. Prove by contradiction that for $X=\{0,1\}, X^{\omega}$ is uncountable. (Hint: suppose there exists a surjective map $f: \mathbb{N} \rightarrow X^{\omega}$, and find an element in $X^{\omega}$ which is not in the image of $f)$.
4. (Dynkins's $\pi-\lambda$ system Theorem): The goal of this exercise is to prove this theorem, and review/practice some of the set theoretical results.

Suppose $\Omega$ is some arbitrary set (which need not have any topological or algebraic structure). Then, $\mathscr{F} \subset 2^{\Omega}$ as a collection of subsets of $\Omega$ is called a $\sigma$-algebra if it satisfies following properties:

- $\emptyset \in \mathscr{F}$.
- For every $A \subset \Omega$ where $A \in \mathscr{F}, A^{c} \in \mathscr{F}\left(A^{c}\right.$ refers to the complement of set $A$, i.e $\Omega \backslash A$ ).
- For every countable sequence of subsets $\left\{A_{n}\right\}$ where $A_{n} \in \mathscr{F}$ for all $n, \bigcup_{n} A_{n} \in$ $\mathscr{F}$.
(a) Show that $\Omega \in \mathscr{F}$.
(b) Prove that $\mathscr{F}$ is closed under countable intersection.

Two more definitions: first, $\Lambda \subset 2^{\Omega}$ is called a $\lambda$-system if:

- $\Omega \in \Lambda$.
- If $A, B \in \Lambda$ and $A \subset B$, then $B \backslash A \in \Lambda$.
- If $\left\{A_{n}\right\}$ is an increasing sequence of subsets, i.e $A_{1} \subset A_{2} \subset \ldots$, with each element being in $\Lambda$, then $\bigcup_{n} A_{n} \in \Lambda$.

Second, $\Pi \subset 2^{\Omega}$ is called a $\pi$-system, if it is closed under finite intersection. Now assume $\Pi$ is a $\pi$-system such that $\Pi \subset \Lambda$, where $\Lambda$ is a $\lambda$-system. The Dynkin's theorem which we want to prove states that the smallest $\sigma$-algebra containing $\Pi$ (denoted by $\sigma(\Pi)$ ) is a subset of $\Lambda$. Try to keep on with each step below until the final result drops out:

[^0](c) Let $\lambda(\Pi)$ be the smallest ${ }^{2} \lambda$-system containing $\Pi$. Explain why $\lambda(\Pi) \subset \Lambda$. (Hint: note that $\Pi \subset \Lambda$ ).
(d) Let $B \in \Pi$ and define $\mathcal{A}_{B}:=\{A \subset \Omega: A \cap B \in \lambda(\Pi)\}$. Show that $\mathcal{A}_{B}$ is itself a $\lambda$-system and contains $\lambda(\Pi)$, i.e $\lambda(\Pi) \subset \mathcal{A}_{B}$.
(e) Now let $A \in \lambda(\Pi)$, and define $\mathcal{B}_{A}:=\{B \subset \Omega: A \cap B \in \lambda(\Pi)\}$. Again, show that $\mathcal{B}_{A}$ is a $\lambda$-system containing $\lambda(\Pi)$.
(f) Use the previous two steps to show $\lambda(\Pi)$ is also a $\pi$-system.
(g) Given that we have seen so far that $\lambda(\Pi)$ is both a $\pi$-system and a $\lambda$-system, deduce that it has to be a $\sigma$-algebra.
(h) Conclude that $\sigma(\Pi) \subset \Lambda$.
5. Let $\mathcal{U}$ and $\mathcal{Z}$ be two sets, and $(u, z) \mapsto P(u, z)$ be a bounded function on $\mathcal{U} \times \mathcal{Z}$. Define the upper and lower value functions as:
\[

$$
\begin{align*}
& V_{+}=\inf _{u \in \mathcal{U}} \sup _{z \in \mathcal{Z}} P(u, z) \\
& V_{-}=\sup _{z \in \mathcal{Z}} \inf _{u \in \mathcal{U}} P(u, z) \tag{1}
\end{align*}
$$
\]

(a) Show that $V_{+} \geq V_{-}$.
(b) Call any function $\beta: \mathcal{U} \rightarrow \mathcal{Z}$ a strategy for the maximizing side. Denote the space of all such strategies as $\mathscr{B}$. Prove the following identity, and explain why it is not in contrast with part (a).

$$
\begin{equation*}
V_{+}=\inf _{u \in \mathcal{U}} \sup _{\beta \in \mathscr{B}} P(u, \beta(u))=\sup _{\beta \in \mathscr{B}} \inf _{u \in \mathcal{U}} P(u, \beta(u)) \tag{2}
\end{equation*}
$$

6. Let $\left\{a_{n}\right\}$ and $\left\{b_{n}\right\}$ be two sequences of non-negative real numbers. Further, let $\left\{s_{n}\right\}$ be a monotone positive sequence satisfying $s_{n} \rightarrow \infty$. Prove that

$$
\begin{equation*}
\limsup _{n \rightarrow \infty}\left(a_{n}^{s_{n}}+b_{n}^{s_{n}}\right)^{1 / s_{n}}=\max \left\{\limsup _{n \rightarrow \infty} a_{n}, \limsup _{n \rightarrow \infty} b_{n}\right\} . \tag{3}
\end{equation*}
$$

(Hint: recall that $\lim \sup _{n \rightarrow \infty} a_{n}=\inf _{m \in \mathbb{N}} \sup _{n \geq m} a_{n}$ )

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[^0]:    ${ }^{1}$ Please keep your answers short and concise. The solution to each (sub)question could well fit in at most one page.

[^1]:    ${ }^{2}$ By smallest we mean: $\lambda(\Pi)=\bigcap_{\left\{\Lambda_{\alpha} \text { is } \lambda \text {-system in } 2^{\Omega}: \Pi \subset \Lambda_{\alpha}\right\}} \Lambda_{\alpha}$

