Econ 204 – Problem Set 2

Due Tuesday, July 31

1. Give an example of a complete metric space which is homeomorphic to an incomplete metric space.

2. Suppose the metric space (X, d) is infinite.

   (a) Show that there exists an open set U such that U and U^c are both infinite.

   (b) Show that there exists an infinite subset $Y \subset X$ such that (Y, d) is a discrete metric space.[1]

3. Prove that the function $f : (0, 1) \to [0, 1)$ given by

$$f(x) = \begin{cases} \frac{1}{q} & x = \frac{p}{q} \in \mathbb{Q} \text{ (in lowest terms)} \\ 0 & x \notin \mathbb{Q} \end{cases}$$

   is continuous at all irrationals and discontinuous at all rationals.

4. Take any mapping f from a metric space X into a metric space Y. Prove that f is continuous if and only if $f\left(\overline{A}\right) \subseteq \overline{f(A)}$ for every set A. (Hint: use the closed set characterization of continuity).

5. Prove that a metric space (X, d) is discrete if and only if every function on X into any other metric space (Y, ρ), where Y has at least two distinct elements, is continuous.

6. Suppose T is an operator on a complete metric space (X, d). Prove that the condition

$$d(T(x), T(y)) < d(x, y) \;\; \forall x, y \in X \; (x \neq y)$$

   does not guarantee the existence of a fixed point of T.

[1] A metric space (X, d) *discrete* if every subset $A \subset X$ is open. Notice that any set equipped with the discrete metric forms a discrete metric space, but not every discrete metric space necessarily has the discrete metric.