1. Give an example of a complete metric space which is homeomorphic to an incomplete metric space.

2. Suppose the metric space \((X, d)\) is infinite.
   
   (a) Show that there exists an open set \(U\) such that \(U\) and \(U^c\) are both infinite.
   
   (b) Show that there exists an infinite subset \(Y \subset X\) such that \((Y, d)\) is a discrete metric space.¹

3. Prove that the function \(f : (0, 1) \to [0, 1)\) given by
   
   \[
   f(x) = \begin{cases} 
   \frac{1}{q} & x = \frac{p}{q} \in \mathbb{Q} \text{ (in lowest terms)} \\
   0 & x \notin \mathbb{Q}
   \end{cases}
   \]

   is continuous at all irrationals and discontinuous at all rationals.

4. Take any mapping \(f\) from a metric space \(X\) into a metric space \(Y\). Prove that \(f\) is continuous if and only if \(f(A) \subseteq f(A)\) for every set \(A\). (Hint: use the closed set characterization of continuity).

5. Prove that a metric space \((X, d)\) is discrete if and only if every function on \(X\) into any other metric space \((Y, \rho)\), where \(Y\) has at least two distinct elements, is continuous.

6. Suppose \(T\) is an operator on a complete metric space \((X, d)\). Prove that the condition
   
   \[d(T(x), T(y)) < d(x, y) \forall x, y \in X \ (x \neq y)\]

   does not guarantee the existence of a fixed point of \(T\).

¹A metric space \((X, d)\) discrete if every subset \(A \subset X\) is open. Notice that any set equipped with the discrete metric forms a discrete metric space, but not every discrete metric space necessarily has the discrete metric.