Econ 204 – Problem Set 3

Due Friday August 3, 2018 [1]

1. Let (X, d) be a metric space:

   (a) Let $y \in X$ be given. Define the function $d_y : X \to \mathbb{R}$ by

   $$d_y(x) = d(x, y) \tag{1}$$

   Show that d_y is a continuous function on X for each $y \in X$.

   (b) Let A be a subset of X and $x \in X$. Recall that the distance from the point x to the set A is defined as:

   $$\rho(x, A) = \inf \{d(x, a) : a \in A\} \tag{2}$$

   Show that the closure of set A is the set of all points with zero distance to A, that is:

   $$\bar{A} = \{x \in X : \rho(x, A) = 0\} \tag{3}$$

   (c) Now let $A \subset X$ be a compact subset. Show that $\rho(x, A) = d(x, a)$ for some $a \in A$.

2. Let x and y be moving objects in $\mathbb{R}$. Time is discrete, namely $t \in \mathbb{Z}_+ := \{0\} \cup \mathbb{N}$. In addition, $\beta > 1$ is a fixed parameter. For $a, b \in \mathbb{R}$, let $\rho(a, b) := |a - b| \wedge 1$ (as mentioned in the section, the symbol $\wedge$ is sometimes used to refer to the minimum of two elements). Then for any $x, y \in \mathbb{R}^\omega$ [2], let

   $$d(x, y) = \sum_{t \in \mathbb{Z}_+} \beta^{-t} \rho(x_t, y_t) \tag{4}$$

   denotes the distance between $x = (x_0, x_1, \ldots)$ and $y = (y_0, y_1, \ldots)$, where x_t is the position of x at time t on the real line.

   (a) Show that d is a metric on $\mathbb{R}^\omega$.

   (b) Show that $(\mathbb{R}^\omega, d)$ is a bounded metric space.

   (c) Is $[0, 1]^\omega$ an open or closed subset of $\mathbb{R}^\omega$? (in either case present a proof)

   (d) Is $(\mathbb{R}^\omega, d)$ a complete metric space? (prove if yes, otherwise provide a counterexample)

   (e) Is $[0, 1]^\omega$ a totally bounded subset under d? Is it a compact subset?

3. Let D be the space of all functions $f : [0, 1] \to \mathbb{R}$ such that f is continuous and such that for some $\varepsilon > 0$, $f : (-\varepsilon, 1 + \varepsilon) \to \mathbb{R}$ is differentiable and $f' : (-\varepsilon, 1 + \varepsilon) \to \mathbb{R}$ is

[1] Please keep your answers short and concise. The solution to each (sub)question could well fit in at most one page.

[2] For the definition of $\mathbb{R}^\omega$ please refer to Q3 of ps.1

continuous. For each $f \in D$, let

$$\|f\|_\infty = \sup\left\{\left|f(t)\right| : t \in [0,1]\right\} \quad \text{and} \quad \left\|f'\right\|_\infty = \sup\left\{\left|f'(t)\right| : t \in [0,1]\right\}. \tag{5}$$

Define the function $\|\cdot\| : D \to \mathbb{R}_+$ by

$$\|f\| := \|f\|_\infty + \left\|f'\right\|_\infty \tag{6}$$

(a) Show that $\left(D, \|\cdot\|\right)$ is a normed vector space.

(b) Define the function $J : D \to \mathbb{R}$ as

$$J(f) = \int_0^1 c(x)\left(f'(x)\right)^2 \mathrm{d}x, \tag{7}$$

where $c : [0,1] \to \mathbb{R}_+$ is a nonnegative integrable function (its integral over $[0,1]$ is finite). Prove that J is continuous.

4. Let X be a connected space. Let $\mathcal{R}$ be an equivalence relation on X such that for each $x \in X$, there exists an open set $\mathcal{O}_x$ containing x such that $\mathcal{O}_x \subset [x]$. Show that $\mathcal{R}$ has only one (distinct) equivalence class.

5. Define the correspondence $\Gamma : [0,1] \to 2^{[0,1]}$ by:

$$\Gamma(x) = \begin{cases} [0,1] \cap \mathbb{Q} & \text{if } x \in [0,1] \setminus \mathbb{Q} \\ [0,1] \setminus \mathbb{Q} & \text{if } x \in [0,1] \cap \mathbb{Q} \end{cases}. \tag{8}$$

Show that Γ is not continuous, but it is lower-hemicontinuous. Is Γ upper-hemicontinuous at any rational? At any irrational? Does this correspondence have a closed graph?

6. Let X be a metric space, and $I : X \to \mathbb{R}_+$ be a lower semi-continuous function [3].

(a) Prove that for every given $\varepsilon > 0$ there exists an open set U_ε containing $x \in X$ such that

$$\inf\{I(y) : y \in U_\varepsilon\} \geq I(x) - \varepsilon. \tag{9}$$

(b) Let $x \in X$. For each $n \in \mathbb{N}$ let

$$m_n = \inf\left\{I(y) : y \in B_{1/n}(x)\right\}. \tag{10}$$

Show that $\{m_n\}$ is an increasing sequence and that $m_n \to I(x)$.

[3] A function $I : X \to \mathbb{R}$ is called lower semi-continuous *iff* for every α the set $\{x : I(x) > \alpha\}$ is open in X.