Econ 204 – Problem Set 5[1]

Due Friday August 10, 2018

1. Assume that $f : [0, \infty) \to \mathbb{R}$ is differentiable for all $x > 0$, and $f'(x) \to 0$ as $x \to \infty$. Prove
$$\lim_{x\to\infty} [f(x+1) - f(x)] = 0. \tag{1}$$
Hint: Use the mean value theorem, and then send $x \to \infty$.

2. Let $f_n : \mathbb{R} \to \mathbb{R}$ be differentiable for each $n \in \mathbb{N}$ with $\left|f_n'(x)\right| \leq 1$ for all n and x. Assume,
$$\lim_{n\to\infty} f_n(x) = g(x) \tag{2}$$
for all x. Prove that $g : \mathbb{R} \to \mathbb{R}$ is Lipschitz-continuous.

3. Let $f : \mathbb{R} \to \mathbb{R}$ be a function that is strictly increasing and twice differentiable, with $f''(x) \geq 0$ for each $x \in \mathbb{R}$. Assume there exists $y \in \mathbb{R}$ such that $f(y) = 0$.
   (a) Show that $f'(x) > 0$ for all $x \in \mathbb{R}$.
   (b) Fix $x_0 > y$ and define the sequence $\{x_n\}$ generated recursively by
$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}. \tag{3}$$
   Show that $x_n \to y$.

4. The goal of this exercise is to verify the **Banach-Steinhaus** theorem. Let $\{T_n\}$ be a sequence of bounded linear functions $T_n : X \to Y$ from a Banach (complete normed vector) space X into a normed vector space Y, such that $\{T_n(x)\}$ is bounded for every $x \in X$, that is for all $x \in X$ there exists $c_x \in \mathbb{R}_+$ such that:
$$\left\|T_n(x)\right\| \leq c_x \quad \forall n \in \mathbb{N} \tag{4}$$
Then, we want to show that the sequence of norms $\left\{\|T_n\|\right\}$ is bounded, that is there exists $c > 0$ such that $\|T_n\| \leq c$ for all $n \in \mathbb{N}$.
   (a) For every $k \in \mathbb{N}$ let $A_k \subseteq X$ be the set of all $x \in X$ such that $\left\|T_n(x)\right\| \leq k$ for all n. Show that A_k is closed under the X-norm.
   (b) Use equation (4) to show that $X = \bigcup_{k\in\mathbb{N}} A_k$.
   (c) The **Baire's** theorem states that in this case since X is complete, there exists some A_{k_0} that contains an open ball, say $B_\varepsilon(x_0) \subseteq A_{k_0}$. Take this result as given, and prove there exists some constant $c > 0$ such that
$$\|T_n\| \leq c \quad \forall n \in \mathbb{N}. \tag{5}$$

[1] Please keep your answers short and concise. The solution to each question could well fit in at most one page.

Hint: For every nonzero $x \in X$ there exists $\gamma > 0$ such that $x = \frac{1}{\gamma}(z - x_0)$, where $x_0, z \in B_\varepsilon(x_0)$ and $\gamma > 0$.

5. Suppose $\Psi : X \to 2^Y$ is a correspondence with nonempty and compact values, where $X \subset \mathbb{R}^n, Y \subset \mathbb{R}^m$ for some n, m. Suppose there exists $\beta \in (0, 1)$ such that for all $x, y \in X$,

$$\sup_{w \in \Psi(y)} \inf_{z \in \Psi(x)} \|w - z\| \leq \beta \|x - y\| \,. \tag{6}$$

Show directly from the definition of upper hemicontinuity that Ψ is upper hemicontinuous.