Econ 204 – Problem Set 6

Due Monday, August 13 in Walker's mailbox (Evans 612)

1. Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a C^1 function. Define $F : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ by

$$F(x, \omega) = f(x) + \omega$$

   Show that there is a set $\Omega_0 \subset \mathbb{R}^n$ of Lebesgue measure zero such that, if $\omega \notin \Omega_0$, then for each x_0 satisfying $F(x_0, \omega_0) = 0$ there is an open set U containing x_0, an open set V containing ω_0, and a C^1 function $h : V \to U$ such that for all $\omega \in V$, $x = h(\omega)$ is the unique element of U satisfying $F(x, \omega) = 0$.

2. Define an open half-space as $S = \{y \in \mathbb{R}^n : p \cdot y < c\}$ for some $p \in \mathbb{R}^n$ and $c \in R$. Show that if $A \subsetneq \mathbb{R}^n$ is non-empty, open and convex, then A is equal to the intersection of all open half-spaces containing A.

3. Call a vector $\pi \in \mathbb{R}^n$ a *probability vector* if

$$\sum_{i}^{n} \pi_i = 1 \text{ and } \pi_i \geq 0 \ \forall i$$

   We say there are n states of the world, and π_i is the probability that state i occurs. Suppose there are two traders (trader 1 and trader 2) who each have a set of prior probability distributions (Π_1 and Π_2) which are nonempty, convex, and compact. Call a *trade* a vector $f \in \mathbb{R}^n$, which denotes the net transfer trader 1 receives in each state of the world (and thus $-f$ is the net transfer trader 2 receives in each state of the world). A trade is *agreeable* if

$$\inf_{\pi \in \Pi_1} \sum_{i=1}^{n} \pi_i f_i > 0 \quad \text{and} \quad \inf_{\pi \in \Pi_2} \sum_{i=1}^{n} \pi_i (-f_i) > 0$$

   Prove that there exists an agreeable trade if and only if there is no common prior (that is, $\Pi_1 \cap \Pi_2 = \varnothing$).

4. Prove the following for any convex set $A \subset \mathbb{R}^n$:

   (a) $\operatorname{int} A$ and $\overline{A}$ are convex.

   (b) If A is closed, then A has a unique smallest element (in terms of norm).

   (c) A is connected.

5. *The Minimax Theorem:* Let $X \subset \mathbb{R}^n$ and $Y \subset \mathbb{R}^m$ be non-empty, compact, and convex sets. Suppose that $f : X \times Y \to \mathbb{R}$ is continuous. Further, for any $x \in X$, $y \in Y$, and $\alpha \in \mathbb{R}$, suppose the sets

$$\{x' \in X : f(x', y) \geq \alpha\}, \quad \{y' \in X : f(x, y') \leq \alpha\}$$

are convex. Show that

$$\max_{x\in X}\min_{y\in Y} f(x,y) = \min_{y\in Y}\max_{x\in X} f(x,y)$$

Hint: consider the following correspondences:

$$\begin{aligned}\Phi(y) &= \arg\max_{x\in X} f(x,y)\\ \Pi(x) &= \arg\min_{y\in Y} f(x,y)\\ \Gamma(x,y) &= \Pi(x)\times\Phi(y)\end{aligned}$$

where $\Phi : Y \to 2^Y$, $\Pi : X \to 2^X$, and $\Gamma : X\times Y \to 2^{X\times Y}$.

6. Consider the following inhomogeneous second-order linear differential equation:

$$x''(t) - 2x'(t) + x(t) = \sin(t)$$

   (a) Find the general solution of the corresponding homogeneous equation.

   (b) Find a particular solution of the original inhomogeneous equation satisfying the initial conditions $x(0) = 1$ and $x'(0) = 0$.

   (c) Find the general solution of the original inhomogeneous equation.