Economics 204 Summer/Fall 2018
Final Exam – Suggested Solutions

Answer all of the questions below. Be as complete, correct, and concise as possible. There are 6 questions for a total of 165 points possible; point values for each problem are in parentheses. For questions with subparts, each subpart is worth the same number of points. You have 180 minutes to complete the exam. Use the points as a guide to allocating your time. You may use any result from class with appropriate references unless you are specifically being asked to prove it.

1. (15) Define or state each of the following.

   (a) convergence of a sequence $\{x_n\}$ to a point x in a metric space (X, d)

   (b) linear transformation between vector spaces X and Y over the same field F

   (c) Separating Hyperplane Theorem

   Solution: See notes.

2. (30) Let $r \in (0, 1)$. Show that for every $n \in \mathbb{N} = \{1, 2, 3, \ldots\}$,

$$\sum_{k=0}^{n} r^k = \frac{1 - r^{n+1}}{1 - r}$$

   Solution: The proof below is by induction. For the base case, let $n = 1$. Then since $r \in (0, 1)$,

$$\sum_{k=0}^{1} r^k = 1 + r = \frac{(1 + r)(1 - r)}{1 - r} = \frac{1 - r^2}{1 - r} = \frac{1 - r^{1+1}}{1 - r}$$

   Thus the claim is true for $n = 1$. For the induction hypothesis, assume that for some $n \geq 1$,

$$\sum_{k=0}^{n} r^k = \frac{1 - r^{n+1}}{1 - r}$$

Now consider $n+1$:

$$\begin{aligned}
\sum_{k=0}^{n+1} r^k &= \sum_{k=0}^{n} r^k + r^{n+1} \\
&= \frac{1-r^{n+1}}{1-r} + r^{n+1} \qquad \text{by the induction hypothesis} \\
&= \frac{1-r^{n+1}+r^{n+1}(1-r)}{1-r} \\
&= \frac{1-r^{n+1}+r^{n+1}-r^{n+2}}{1-r} \\
&= \frac{1-r^{n+2}}{1-r} = \frac{1-r^{(n+1)+1}}{1-r}
\end{aligned}$$

Thus by induction, the claim holds for every $n \in \mathbb{N}$.

3. (30) Let X be a finite-dimensional vector space, and V and W be vector subspaces of X such that $V \cap W = \{0\}$.

(a.) Show that $V + W = \{x \in X : x = v + w \text{ for some } v \in V, w \in W\}$ is a vector subspace of X.

Solution: X is a vector space over the field F. Clearly $V + W \subseteq X$. To show that $V + W$ is a vector subspace of X, it suffices to show that for all $x_1, x_2 \in V + W$ and for all $\alpha, \beta \in F$, $\alpha x_1 + \beta x_2 \in V + W$. Thus let $x_1, x_2 \in V + W$. By definition, $\exists v_1, v_2 \in V, w_1, w_2 \in W$ such that $x_1 = v_1 + w_1$ and $x_2 = v_2 + w_2$. Then

$$\alpha x_1 + \beta x_2 = \alpha(v_1 + w_1) + \beta(v_2 + w_2) = (\alpha v_1 + \beta v_2) + (\alpha w_1 + \beta w_2)$$

Since V and W are vector subspaces of X, $\alpha v_1 + \beta v_2 \in V$ and $\alpha w_1 + \beta w_2 \in W$. Then by definition,

$$\alpha x_1 + \beta x_2 = (\alpha v_1 + \beta v_2) + (\alpha w_1 + \beta w_2) \in V + W$$

Thus $V + W$ is a vector subspace of X.

(b.) Show that $\dim\,(V + W) = \dim V + \dim W$.

Solution: First note that because X is finite-dimensional and V and W are vector subspaces of X, V and W must also be finite-dimensional (otherwise, there is a linearly independent subset of X with cardinality strictly greater than $\dim X$, in particular a basis for V or for W). Then let $\{v_1, \ldots, v_n\}$ be a basis for V and let $\{w_1, \ldots, w_m\}$ be a basis for W. These are finite by the previous observation. Let

$$U = \{v_1, \ldots, v_n, w_1, \ldots, w_m\}$$

Now claim U is a basis for $V + W$. To show this, we must show that U is linearly independent and spans $V + W$.

To see that U spans $V+W$, first note that $U \subseteq V+W$, as $v_i = v_i + 0 \in V+W$ and $w_j = 0 + w_j \in V+W$ for each $i = 1, \ldots, n$ and each $j = 1, \ldots, m$. Then since $V+W$ is a vector subspace of X, span $U \subseteq V+W$. Now let $x \in V+W$. By definition, $\exists v \in V, w \in W$ such that $x = v + w$. Then

$$v = \sum_{i=1}^{n} \alpha_i v_i \text{ and } w = \sum_{j=1}^{m} \beta_j w_j \quad \text{for some } \alpha_i, \beta_j \in F, i = 1, \ldots, n, j = 1, \ldots, m$$

since $\{v_1, \ldots, v_n\}$ is a basis for V and $\{w_1, \ldots, w_m\}$ is a basis for W. Then

$$x = v + w = \sum_{i=1}^{n} \alpha_i v_i + \sum_{j=1}^{m} \beta_j w_j$$

So $x \in \text{span } U$. Since $x \in V+W$ was arbitrary, $V+W \subseteq \text{span } U$. Thus $V+W = \text{span } U$.

To see that U is linearly independent, suppose

$$\sum_{i=1}^{n} \alpha_i v_i + \sum_{j=1}^{m} \beta_j w_j = 0 \quad \text{for some } \alpha_i, \beta_j \in F, i = 1, \ldots, n, j = 1, \ldots, m$$

Then

$$\sum_{i=1}^{n} \alpha_i v_i = -\sum_{j=1}^{m} \beta_j w_j$$

and since V and W are vector subspaces, $\sum_{i=1}^{n} \alpha_i v_i \in V$ and $-\sum_{j=1}^{m} \beta_j w_j \in W$. Thus

$$\sum_{i=1}^{n} \alpha_i v_i = -\sum_{j=1}^{m} \beta_j w_j \in V \cap W$$

By assumption $V \cap W = \{0\}$, so this implies $\sum_{i=1}^{n} \alpha_i v_i = -\sum_{j=1}^{m} \beta_j w_j = 0$. Since $\{v_1, \ldots v_n\}$ is a basis for V it is linearly independent, which implies $\alpha_i = 0$ for each i; similarly, since $\{w_1, \ldots, w_m\}$ is a basis for W it is linearly independent, which implies $\beta_j = 0$ for each j. Thus U is linearly independent.

Thus U is a basis for $V+W$, and by construction U has cardinality $n+m$. By definition, the dimension of a vector space is the cardinality of any basis, thus

$$\dim (V+W) = n + m = \dim V + \dim W$$

4. (30) Let (X, d) be a metric space and $C \subseteq X$ be compact. Let $A \subseteq C$ be a nonempty set. Suppose that for each $x \in C$ there exists $\varepsilon_x > 0$ such that $A \cap B_{\varepsilon_x}(x)$ is either empty or is a finite set. Show that A is a finite set.

Solution: For each $x \in C$, let $\varepsilon_x > 0$ be such that $A \cap B_{\varepsilon_x}(x)$ is either empty or finite. The collection $\{B_{\varepsilon_x}(x) : x \in C\}$ is an open cover of C, as each open ball $B_{\varepsilon_x}(x)$ is an open set, and by definition, $x \in B_{\varepsilon_x}(x)$ for each $x \in C$. Since C is compact, there exist $x_1, \ldots, x_n \in C$ such that

$$C \subseteq B_{\varepsilon_{x_1}}(x_1) \cup \cdots \cup B_{\varepsilon_{x_n}}(x_n)$$

Since $A \subseteq C$, this implies

$$A = A \cap C \subseteq \bigcup_{i=1}^{n} \left(A \cap B_{\varepsilon_{x_i}}(x_i)\right)$$

For each i, $A \cap B_{\varepsilon_{x_i}}(x_i)$ is either empty or finite, thus $\cup_{i=1}^{n}(A \cap B_{\varepsilon_{x_i}}(x_i))$ is either empty or finite, as it is a finite union of sets, each of which is either empty or finite. But $A \neq \emptyset$ by assumption, and $A \subseteq \cup_{i=1}^{n}(A \cap B_{\varepsilon_{x_i}}(x_i))$, thus $\cup_{i=1}^{n}(A \cap B_{\varepsilon_{x_i}}(x_i))$ must be a finite set. This implies A is a finite set.

5. (30) Let $f : [a, b] \to [a, b]$ where $a, b \in \mathbb{R}$ and $a < b$. Suppose f is continuous on $[a, b]$ and differentiable on (a, b). Suppose f has a fixed point x^* such that $x^* \in (a, b)$ and $f'(x^*) > 1$. Show that f has at least three fixed points.

Solution: Let $g : [a, b] \to \mathbb{R}$ be given by

$$g(x) = f(x) - x$$

Then note that g is continuous on $[a, b]$ and differentiable on (a, b), with

$$g'(x) = f'(x) - 1 \quad \forall x \in (a, b)$$

Also note that $x \in [a, b]$ is a fixed point of f if and only if $g(x) = 0$.

Then by assumption $x^* \in (a, b)$ is a fixed point of f, which implies $g(x^*) = 0$. Also $f'(x^*) > 1$, which implies $g'(x^*) = f'(x^*) - 1 > 0$. Then by definition,

$$g'(x^*) = \lim_{h \to 0} \frac{g(x^* + h) - g(x^*)}{h} = \lim_{h \to 0} \frac{g(x^* + h)}{h}$$

where the second equality uses the fact that $g(x^*) = 0$. Now since $g'(x^*) > 0$, this implies there exists $\varepsilon > 0$ such that for all $|h| < \varepsilon, h \neq 0$,

$$\frac{g(x^* + h)}{h} > 0$$

Then for $h \in (0, \varepsilon)$,

$$\frac{g(x^* + h)}{h} > 0 \Rightarrow g(x^* + h) > 0$$

and for $h \in (-\varepsilon, 0)$,

$$\frac{g(x^* + h)}{h} > 0 \Rightarrow g(x^* + h) < 0$$

So there exists $\varepsilon > 0$ such that

$$g(x) > 0 \quad \forall x \in (x^*, x^* + \varepsilon)$$

and

$$g(x) < 0 \;\; \forall x \in (x^* - \varepsilon, x^*)$$

Now claim that there exists $x_1 \in [a, x^*)$ and $x_2 \in (x^*, b]$ such that

$$g(x_1) = 0 = g(x_2)$$

To see this, consider $[a, x^*)$ first. If $g(a) = 0$, we are done. Otherwise, since $f(a) \geq a$,

$$g(a) = f(a) - a > 0$$

Then there exists $x \in (x^* - \varepsilon, x^*)$ such that

$$g(x) < 0$$

By the Intermediate Value Theorem, there exists $x_1 \in (a, x)$ such that

$$g(x_1) = 0$$

Similarly, consider $(x^*, b]$. If $g(b) = 0$, we are done. Otherwise, since $f(b) \leq b$,

$$g(b) = f(b) - b < 0$$

Then there exists $\bar{x} \in (x^*, x^* + \varepsilon)$ such that

$$g(\bar{x}) > 0$$

By the Intermediate Value Theorem, there exists $x_2 \in (\bar{x}, b)$ such that

$$g(x_2) = 0$$

Then $x_1 \in [a, x^*)$ and $x_2 \in (x^*, b]$, so $x_1 \neq x^*$, $x^* \neq x_2$, and $x_1 \neq x_2$. Since $g(x_1) = g(x_2) = 0$,

$$g(x_1) = f(x_1) - x_1 = 0 \Rightarrow f(x_1) = x_1$$

and

$$g(x_2) = f(x_2) - x_2 = 0 \Rightarrow f(x_2) = x_2$$

Thus x_1, x_2, and x^* are all fixed points of f, which shows that f has at least three fixed points.

6. (30) Let $X \subseteq \mathbb{R}^n$ and $\Psi : X \to 2^{\mathbb{R}}$ be an upper hemicontinuous correspondence with nonempty, compact values, so $\Psi(x) \subseteq \mathbb{R}$ is a nonempty compact set for each $x \in X$. Let $C \subseteq X$ be compact. Show that Ψ attains its maximum and minimum on C. That is, if

$$M = \sup\{y \in \mathbb{R} : y \in \Psi(x) \text{ for some } x \in C\} \quad m = \inf\{y \in \mathbb{R} : y \in \Psi(x) \text{ for some } x \in C\}$$

then show that there exist $x_M, x_m \in C$ such that $M \in \Psi(x_M)$ and $m \in \Psi(x_m)$.

Solution: The solution will show that there exists $x_M \in C$ such that $M \in \Psi(x_M)$; the proof for the infimum m is analogous.

If $M < +\infty$, then for every $n \in \mathbb{N}$ choose $x_n \in C$ and $y_n \in \Psi(x_n)$ such that

$$M - \frac{1}{n} \leq y_n \leq M$$

If $M = +\infty$, then for every $n \in \mathbb{N}$ choose $x_n \in C$ and $y_n \in \Psi(x_n)$ such that

$$y_n > n$$

In either case, $\{x_n\} \subseteq C$ and $y_n \in \Psi(x_n)$ for each n. Also by construction, in either case $y_n \to M$.

Since C is compact, there is a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \to x \in C$. Since $\Psi(x)$ is compact it is bounded, which implies there is a bounded open set $V \supseteq \Psi(x)$. Since Ψ is uhc at x, there exists an open set $U \ni x$ such that for all $x' \in U$, $\Psi(x') \subseteq V$. Then $x_{n_k} \to x \Rightarrow \exists K$ such that for all $k > K$, $x_{n_k} \in U$. Thus

$$\Psi(x_{n_k}) \subseteq V \quad \forall k > K$$

Since $y_{n_k} \in \Psi(x_{n_k})$ $\forall k$, $y_{n_k} \in V$ for all $k > K$. Since V is bounded, this implies

$$M < +\infty \text{ and } \lim_k y_{n_k} = M$$

Now claim $M \in \Psi(x)$. To see this, first I will give an argument using the definition of uhc. For this argument, suppose by way of contradiction that $M \notin \Psi(x)$. Since $\Psi(x)$ is compact, there exist open sets V', V'' such that $V' \supseteq \Psi(x)$, $V'' \ni M$, and $V' \cap V'' = \emptyset$.[1] Since Ψ is uhc at x, there exists an open set $U' \ni x$ such that for all $x' \in U'$, $\Psi(x') \subseteq V'$. Then there exists K' such that for all $k > K'$, $x_{n_k} \in U'$. Thus

$$\Psi(x_{n_k}) \subseteq V' \quad \forall k > K'$$

Thus $y_{n_k} \in V'$ for all $k > K'$. Since $V' \cap V'' = \emptyset$, this implies $y_{n_k} \notin V''$ for all $k > K'$. But $y_{n_k} \to M$, which is a contradiction. Thus $M \in \Psi(x)$.

Here is an argument using the fact that Ψ must have closed graph under the assumptions of the problem. Note that Ψ is uhc and compact-valued, so it is uhc and closed-valued. Thus Ψ has closed graph. By the above argument $M \in \mathbb{R}$ (that is, $M < +\infty$), and $x_{n_k} \to x \in X$, $y_{n_k} \to M \in \mathbb{R}$ with $y_{n_k} \in \Psi(x_{n_k})$ for each n_k. Thus $(x_{n_k}, y_{n_k}) \in$ graph Ψ for all k, and $(x_{n_k}, y_{n_k}) \to (x, M)$. Since graph Ψ is closed, $(x, M) \in$ graph Ψ, that is, $M \in \Psi(x)$.

Here is an argument using the sequential characterization of uhc. By the construction above, $x_{n_k} \to x$ and $y_{n_k} \in \Psi(x_{n_k})$ for each k. Since Ψ is uhc and compact-valued,

[1] For example, using the fact that $\Psi(x)$ is compact and $M \notin \Psi(x)$, $d(M, \Psi(x)) > 0$. Then for $\varepsilon > 0$ such that $\varepsilon < d(M, \Psi(x))/2$, $B_\varepsilon(M) \cap \left(\cup_{y \in \Psi(x)} B_\varepsilon(y)\right) = \emptyset$ and $B_\varepsilon(M)$, $\cup_{y \in \Psi(x)} B_\varepsilon(y)$ are open.

we can use the sequential characterization of uhc to conclude that there is a further subsequence $\{y_{n_{k_\ell}}\}$ of $\{y_{n_k}\}$ such that

$$y_{n_{k_\ell}} \to y \in \Psi(x)$$

for some $y \in \Psi(x)$. But since the parent sequence $y_{n_k} \to M$, the subsequence $y_{n_{k_\ell}} \to M$ as well. By uniqueness of limits, $y = M$. Thus $y = M \in \Psi(x)$. (Note that this argument actually simultaneously establishes that $M < +\infty$ and that $M \in \Psi(x)$.)