1. (15) Define or state each of the following.

(a) convergence of a sequence \( \{x_n\} \) to a point \( x \) in a metric space \((X, d)\)

(b) linear transformation between vector spaces \( X \) and \( Y \) over the same field \( F \)

(c) Separating Hyperplane Theorem

**Solution:** See notes.

2. (30) Let \( r \in (0,1) \). Show that for every \( n \in \mathbb{N} = \{1, 2, 3, \ldots\} \),

\[
\sum_{k=0}^{n} r^k = \frac{1 - r^{n+1}}{1 - r}
\]

**Solution:** The proof below is by induction. For the base case, let \( n = 1 \). Then since \( r \in (0,1) \),

\[
\sum_{k=0}^{1} r^k = 1 + r = \frac{(1 + r)(1 - r)}{1 - r} = \frac{1 - r^2}{1 - r} = \frac{1 - r^{1+1}}{1 - r}
\]

Thus the claim is true for \( n = 1 \). For the induction hypothesis, assume that for some \( n \geq 1 \),

\[
\sum_{k=0}^{n} r^k = \frac{1 - r^{n+1}}{1 - r}
\]
Now consider \( n + 1 \):

\[
\sum_{k=0}^{n+1} r^k = \sum_{k=0}^{n} r^k + r^{n+1} = \frac{1 - r^{n+1}}{1 - r} + r^{n+1} \quad \text{by the induction hypothesis}
\]

\[
= \frac{1 - r^{n+1} + r^{n+1}(1 - r)}{1 - r} = \frac{1 - r^{n+1} + r^{n+1} - r^{n+2}}{1 - r} = \frac{1 - r^{n+2}}{1 - r} = \frac{1 - r^{n(n+1)+1}}{1 - r}
\]

Thus by induction, the claim holds for every \( n \in \mathbb{N} \).

3. (30) Let \( X \) be a finite-dimensional vector space, and \( V \) and \( W \) be vector subspaces of \( X \) such that \( V \cap W = \{0\} \).

(a.) Show that \( V + W = \{ x \in X : x = v + w \text{ for some } v \in V, w \in W \} \) is a vector subspace of \( X \).

\textbf{Solution:} \( X \) is a vector space over the field \( F \). Clearly \( V + W \subseteq X \). To show that \( V + W \) is a vector subspace of \( X \), it suffices to show that for all \( x_1, x_2 \in V + W \) and for all \( \alpha, \beta \in F \), \( \alpha x_1 + \beta x_2 \in V + W \). Thus let \( x_1, x_2 \in V + W \). By definition, \( \exists v_1, v_2 \in V, w_1, w_2 \in W \) such that \( x_1 = v_1 + w_1 \) and \( x_2 = v_2 + w_2 \). Then

\[
\alpha x_1 + \beta x_2 = \alpha(v_1 + w_1) + \beta(v_2 + w_2) = (\alpha v_1 + \beta v_2) + (\alpha w_1 + \beta w_2)
\]

Since \( V \) and \( W \) are vector subspaces of \( X \), \( \alpha v_1 + \beta v_2 \in V \) and \( \alpha w_1 + \beta w_2 \in W \). Then by definition,

\[
\alpha x_1 + \beta x_2 = (\alpha v_1 + \beta v_2) + (\alpha w_1 + \beta w_2) \in V + W
\]

Thus \( V + W \) is a vector subspace of \( X \).

(b.) Show that dim \( (V + W) \) = dim \( V \) + dim \( W \).

\textbf{Solution:} First note that because \( X \) is finite-dimensional and \( V \) and \( W \) are vector subspaces of \( X \), \( V \) and \( W \) must also be finite-dimensional (otherwise, there is a linearly independent subset of \( X \) with cardinality strictly greater than \( \dim X \), in particular a basis for \( V \) or for \( W \)). Then let \( \{v_1, \ldots, v_n\} \) be a basis for \( V \) and \( \{w_1, \ldots, w_m\} \) be a basis for \( W \). These are finite by the previous observation. Let

\[
U = \{v_1, \ldots, v_n, w_1, \ldots, w_m\}
\]

Now claim \( U \) is a basis for \( V + W \). To show this, we must show that \( U \) is linearly independent and spans \( V + W \).
To see that $U$ spans $V + W$, first note that $U \subseteq V + W$, as $v_i = v_i + 0 \in V + W$ and $w_j = 0 + w_j \in V + W$ for each $i = 1, \ldots, n$ and each $j = 1, \ldots, m$. Then since $V + W$ is a vector subspace of $X$, span $U \subseteq V + W$. Now let $x \in V + W$. By definition, $\exists v \in V, w \in W$ such that $x = v + w$. Then

$$v = \sum_{i=1}^{n} \alpha_i v_i \text{ and } w = \sum_{j=1}^{m} \beta_j w_j \text{ for some } \alpha_i, \beta_j \in F, i = 1, \ldots, n, j = 1, \ldots, m$$

since $\{v_1, \ldots, v_n\}$ is a basis for $V$ and $\{w_1, \ldots, w_m\}$ is a basis for $W$. Then

$$x = v + w = \sum_{i=1}^{n} \alpha_i v_i + \sum_{j=1}^{m} \beta_j w_j$$

So $x \in \text{span } U$. Since $x \in V + W$ was arbitrary, $V + W \subseteq \text{span } U$. Thus $V + W = \text{span } U$.

To see that $U$ is linearly independent, suppose

$$\sum_{i=1}^{n} \alpha_i v_i + \sum_{j=1}^{m} \beta_j w_j = 0 \text{ for some } \alpha_i, \beta_j \in F, i = 1, \ldots, n, j = 1, \ldots, m$$

Then

$$\sum_{i=1}^{n} \alpha_i v_i = -\sum_{j=1}^{m} \beta_j w_j$$

and since $V$ and $W$ are vector subspaces, $\sum_{i=1}^{n} \alpha_i v_i \in V$ and $-\sum_{j=1}^{m} \beta_j w_j \in W$.

Thus

$$\sum_{i=1}^{n} \alpha_i v_i = -\sum_{j=1}^{m} \beta_j w_j \in V \cap W$$

By assumption $V \cap W = \{0\}$, so this implies $\sum_{i=1}^{n} \alpha_i v_i = -\sum_{j=1}^{m} \beta_j w_j = 0$. Since $\{v_1, \ldots, v_n\}$ is a basis for $V$ it is linearly independent, which implies $\alpha_i = 0$ for each $i$; similarly, since $\{w_1, \ldots, w_m\}$ is a basis for $W$ it is linearly independent, which implies $\beta_j = 0$ for each $j$. Thus $U$ is linearly independent.

Thus $U$ is a basis for $V + W$, and by construction $U$ has cardinality $n + m$. By definition, the dimension of a vector space is the cardinality of any basis, thus

$$\dim (V + W) = n + m = \dim V + \dim W$$

4. (30) Let $(X, d)$ be a metric space and $C \subseteq X$ be compact. Let $A \subseteq C$ be a nonempty set. Suppose that for each $x \in C$ there exists $\varepsilon_x > 0$ such that $A \cap B_{\varepsilon_x}(x)$ is either empty or is a finite set. Show that $A$ is a finite set.

**Solution:** For each $x \in C$, let $\varepsilon_x > 0$ be such that $A \cap B_{\varepsilon_x}(x)$ is either empty or finite. The collection $\{B_{\varepsilon_x}(x) : x \in C\}$ is an open cover of $C$, as each open ball $B_{\varepsilon_x}(x)$ is an
open set, and by definition, \( x \in B_{\varepsilon_x}(x) \) for each \( x \in C \). Since \( C \) is compact, there exist \( x_1, \ldots, x_n \in C \) such that

\[
C \subseteq B_{\varepsilon_{x_1}}(x_1) \cup \cdots \cup B_{\varepsilon_{x_n}}(x_n)
\]

Since \( A \subseteq C \), this implies

\[
A = A \cap C \subseteq \bigcup_{i=1}^{n} (A \cap B_{\varepsilon_{x_i}}(x_i))
\]

For each \( i \), \( A \cap B_{\varepsilon_{x_i}}(x_i) \) is either empty or finite, thus \( \bigcup_{i=1}^{n} (A \cap B_{\varepsilon_{x_i}}(x_i)) \) is either empty or finite, as it is a finite union of sets, each of which is either empty or finite. But \( A \neq \emptyset \) by assumption, and \( A \subseteq \bigcup_{i=1}^{n} (A \cap B_{\varepsilon_{x_i}}(x_i)) \), thus \( \bigcup_{i=1}^{n} (A \cap B_{\varepsilon_{x_i}}(x_i)) \) must be a finite set. This implies \( A \) is a finite set.

5. (30) Let \( f : [a, b] \to [a, b] \) where \( a, b \in \mathbb{R} \) and \( a < b \). Suppose \( f \) is continuous on \( [a, b] \) and differentiable on \((a, b)\). Suppose \( f \) has a fixed point \( x^* \) such that \( x^* \in (a, b) \) and \( f'(x^*) > 1 \). Show that \( f \) has at least three fixed points.

**Solution:** Let \( g : [a, b] \to \mathbb{R} \) be given by

\[
g(x) = f(x) - x
\]

Then note that \( g \) is continuous on \([a, b]\) and differentiable on \((a, b)\), with

\[
g'(x) = f'(x) - 1 \quad \forall x \in (a, b)
\]

Also note that \( x \in [a, b] \) is a fixed point of \( f \) if and only if \( g(x) = 0 \).

Then by assumption \( x^* \in (a, b) \) is a fixed point of \( f \), which implies \( g(x^*) = 0 \). Also \( f'(x^*) > 1 \), which implies \( g'(x^*) = f'(x^*) - 1 > 0 \). Then by definition,

\[
g'(x^*) = \lim_{h \to 0} \frac{g(x^* + h) - g(x^*)}{h} = \lim_{h \to 0} \frac{g(x^* + h)}{h}
\]

where the second equality uses the fact that \( g(x^*) = 0 \). Now since \( g'(x^*) > 0 \), this implies there exists \( \varepsilon > 0 \) such that for all \( |h| < \varepsilon, h \neq 0 \),

\[
\frac{g(x^* + h)}{h} > 0
\]

Then for \( h \in (0, \varepsilon) \),

\[
\frac{g(x^* + h)}{h} > 0 \Rightarrow g(x^* + h) > 0
\]

and for \( h \in (-\varepsilon, 0) \),

\[
\frac{g(x^* + h)}{h} > 0 \Rightarrow g(x^* + h) < 0
\]

So there exists \( \varepsilon > 0 \) such that

\[
g(x) > 0 \quad \forall x \in (x^*, x^* + \varepsilon)
\]
and 

\[ g(x) < 0 \quad \forall x \in (x^* - \varepsilon, x^*) \]

Now claim that there exists \( x_1 \in [a, x^*] \) and \( x_2 \in (x^*, b] \) such that 

\[ g(x_1) = 0 = g(x_2) \]

To see this, consider \([a, x^*)\) first. If \( g(a) = 0 \), we are done. Otherwise, since \( f(a) \geq a \), 

\[ g(a) = f(a) - a > 0 \]

Then there exists \( x \in (x^* - \varepsilon, x^*) \) such that 

\[ g(x) < 0 \]

By the Intermediate Value Theorem, there exists \( x_1 \in (a, x) \) such that 

\[ g(x_1) = 0 \]

Similarly, consider \((x^*, b]\). If \( g(b) = 0 \), we are done. Otherwise, since \( f(b) \leq b \), 

\[ g(b) = f(b) - b < 0 \]

Then there exists \( \bar{x} \in (x^*, x^* + \varepsilon) \) such that 

\[ g(\bar{x}) > 0 \]

By the Intermediate Value Theorem, there exists \( x_2 \in (\bar{x}, b) \) such that 

\[ g(x_2) = 0 \]

Then \( x_1 \in [a, x^*) \) and \( x_2 \in (x^*, b] \), so \( x_1 \neq x^* \), \( x^* \neq x_2 \), and \( x_1 \neq x_2 \). Since \( g(x_1) = g(x_2) = 0 \), 

\[ g(x_1) = f(x_1) - x_1 = 0 \Rightarrow f(x_1) = x_1 \]

and 

\[ g(x_2) = f(x_2) - x_2 = 0 \Rightarrow f(x_2) = x_2 \]

Thus \( x_1, x_2 \), and \( x^* \) are all fixed points of \( f \), which shows that \( f \) has at least three fixed points.

6. (30) Let \( X \subseteq \mathbb{R}^n \) and \( \Psi : X \rightarrow 2^{\mathbb{R}} \) be an upper hemicontinuous correspondence with nonempty, compact values, so \( \Psi(x) \subseteq \mathbb{R} \) is a nonempty compact set for each \( x \in X \). Let \( C \subseteq X \) be compact. Show that \( \Psi \) attains its maximum and minimum on \( C \). That is, if 

\[
M = \sup \{ y \in \mathbb{R} : y \in \Psi(x) \text{ for some } x \in C \} \quad m = \inf \{ y \in \mathbb{R} : y \in \Psi(x) \text{ for some } x \in C \}
\]

then show that there exist \( x_M, x_m \in C \) such that \( M \in \Psi(x_M) \) and \( m \in \Psi(x_m) \).
Solution: The solution will show that there exists \( x_M \in C \) such that \( M \in \Psi(x_M) \); the proof for the infimum \( m \) is analogous.

If \( M < +\infty \), then for every \( n \in \mathbb{N} \) choose \( x_n \in C \) and \( y_n \in \Psi(x_n) \) such that

\[
M - \frac{1}{n} \leq y_n \leq M
\]

If \( M = +\infty \), then for every \( n \in \mathbb{N} \) choose \( x_n \in C \) and \( y_n \in \Psi(x_n) \) such that

\[
y_n > n
\]

In either case, \( \{x_n\} \subseteq C \) and \( y_n \in \Psi(x_n) \) for each \( n \). Also by construction, in either case \( y_n \to M \).

Since \( C \) is compact, there is a subsequence \( \{x_{n_k}\} \) of \( \{x_n\} \) such that \( x_{n_k} \to x \in C \). Since \( \Psi(x) \) is compact it is bounded, which implies there is a bounded open set \( V \supseteq \Psi(x) \). Since \( \Psi \) is uhc at \( x \), there exists an open set \( U \ni x \) such that for all \( x' \in U \), \( \Psi(x') \subseteq V \).

Then \( x_{n_k} \to x \Rightarrow \exists K \) such that for all \( k > K \), \( x_{n_k} \in U \). Thus

\[
\Psi(x_{n_k}) \subseteq V \quad \forall k > K
\]

Since \( y_{n_k} \in \Psi(x_{n_k}) \forall k, y_{n_k} \in V \) for all \( k > K \). Since \( V \) is bounded, this implies

\[
M < +\infty \text{ and } \lim_{k} y_{n_k} = M
\]

Now claim \( M \in \Psi(x) \). To see this, first I will give an argument using the definition of uhc. For this argument, suppose by way of contradiction that \( M \notin \Psi(x) \). Since \( \Psi(x) \) is compact, there exist open sets \( V', V'' \) such that \( V' \supseteq \Psi(x), V'' \supseteq M \), and \( V' \cap V'' = \emptyset \). Since \( \Psi \) is uhc at \( x \), there exists an open set \( U' \ni x \) such that for all \( x' \in U' \), \( \Psi(x') \subseteq V' \). Then there exists \( K' \) such that for all \( k > K' \), \( x_{n_k} \in U' \). Thus

\[
\Psi(x_{n_k}) \subseteq V' \quad \forall k > K'
\]

Thus \( y_{n_k} \in V' \) for all \( k > K' \). Since \( V' \cap V'' = \emptyset \), this implies \( y_{n_k} \notin V'' \) for all \( k > K' \). But \( y_{n_k} \to M \), which is a contradiction. Thus \( M \in \Psi(x) \).

Here is an argument using the sequential characterization of uhc. By the construction above, \( x_{n_k} \to x \) and \( y_{n_k} \in \Psi(x_{n_k}) \) for each \( k \). Since \( \Psi \) is uhc and compact-valued,

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\footnote{For example, using the fact that \( \Psi(x) \) is compact and \( M \notin \Psi(x) \), \( d(M, \Psi(x)) > 0 \). Then for \( \varepsilon > 0 \) such that \( \varepsilon < d(M, \Psi(x))/2 \), \( B_{\varepsilon}(M) \cap (\cup_{y \in \Psi(x)} B_{\varepsilon}(y)) = \emptyset \) and \( B_{\varepsilon}(M) \cup _{y \in \Psi(x)} B_{\varepsilon}(y) \) are open.}

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we can use the sequential characterization of uhc to conclude that there is a further subsequence \( \{ y_{n_k} \} \) of \( \{ y_n \} \) such that

\[
y_{n_k} \to y \in \Psi(x)
\]

for some \( y \in \Psi(x) \). But since the parent sequence \( y_{n_k} \to M \), the subsequence \( y_{n_k} \to M \) as well. By uniqueness of limits, \( y = M \). Thus \( y = M \in \Psi(x) \). (Note that this argument actually simultaneously establishes that \( M < +\infty \) and that \( M \in \Psi(x) \).)