1. Diagonalization of Real Symmetric Matrices
2. Application to Quadratic Forms
3. Linear Maps Between Normed Spaces
How Might This Matter

- Why does diagonalizability matter?

Consider a two-dimensional linear difference equation:

\[
\begin{pmatrix}
  c_{t+1} \\
  k_{t+1}
\end{pmatrix}
= \begin{pmatrix}
  b_{11} & b_{12} \\
  b_{21} & b_{22}
\end{pmatrix}
\begin{pmatrix}
  c_t \\
  k_t
\end{pmatrix}
\quad \forall t = 0, 1, 2, 3, \ldots
\]

given an initial condition \( c_0, k_0 \), or, setting

\[
y_t = \begin{pmatrix}
  c_t \\
  k_t
\end{pmatrix}
\quad \forall t
\]

and

\[
B = \begin{pmatrix}
  b_{11} & b_{12} \\
  b_{21} & b_{22}
\end{pmatrix}
\]

we can rewrite this more compactly as

\[
y_{t+1} = By_t \quad \forall t
\]

where \( b_{ij} \in \mathbb{R} \) each \( i, j \).
We want to find a solution $y_t$, $t = 1, 2, 3, \ldots$ given initial condition $y_0$. (Why?)

Such a dynamical system will arise for example as a characterization of the solution to a standard infinite-horizon optimal growth problem (202a, lecture 2).

If $B$ is diagonalizable, this can be easily solved after a change of basis. If $B$ is diagonalizable, choose an invertible $2 \times 2$ real matrix $P$ such that

$$P^{-1}BP = D = \begin{pmatrix} d_1 & 0 \\ 0 & d_2 \end{pmatrix}$$

Then

$$y_{t+1} = By_t \quad \forall t \iff P^{-1}y_{t+1} = P^{-1}By_t \quad \forall t$$

$$\iff [P^{-1}y_{t+1}] = (P^{-1}BP)P^{-1}y_t \quad \forall t$$

$$\iff \bar{y}_{t+1} = D\bar{y}_t \quad \forall t$$

$$= \begin{pmatrix} 0 & d_2 \\ d_1 & 0 \end{pmatrix} \bar{y}_t$$

where $\bar{y}_t = P^{-1}y_t \quad \forall t$
where $\bar{y}_t = P^{-1}y_t \ \forall t$.

Since $D$ is diagonal, after a change of basis to $\bar{y}_t$, we need to solve two independent linear univariate difference equations, which is easy:

$$\bar{y}_{it} = d_i^t \bar{y}_{i0} \ \forall t$$

- Not all real $n \times n$ matrices are diagonalizable (not even all invertible $n \times n$ matrices are)...so can we identify some classes that are? yesterday: basis of eigenvectors ($\Rightarrow$) n distinct eigenvalues ($\Rightarrow$)

- Some types of matrices appear more frequently than others – especially real symmetric $n \times n$ matrices (matrix representation of second derivatives of $C^2$ functions, quadratic forms...). e.g. second order conditions in optimization, checking concavity and convexity, Taylor series approximation of function
• Recall that an $n \times n$ real matrix $A$ is symmetric if $a_{ij} = a_{ji}$ for all $i, j$, where $a_{ij}$ is the $(i,j)^{th}$ entry of $A$. 
Orthonormal Bases

Definition 1. Let

\[ \delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases} \]

A basis \( V = \{v_1, \ldots, v_n\} \) of \( \mathbb{R}^n \) is orthonormal if \( v_i \cdot v_j = \delta_{ij} \).

In other words, a basis is orthonormal if each basis element has unit length (\( \|v_i\|^2 = v_i \cdot v_i = 1 \ \forall i \)), and distinct basis elements are perpendicular (\( v_i \cdot v_j = 0 \) for \( i \neq j \)).

\[ \|x\| = \left( \sum_{i=1}^{n} x_i^2 \right)^{\frac{1}{2}} = (x \cdot x)^{\frac{1}{2}} \]
Orthonormal Bases

**Remark:** Suppose that \( x = \sum_{j=1}^{n} \alpha_j v_j \) where \{\( v_1, \ldots, v_n \)\} is an orthonormal basis of \( \mathbb{R}^n \). Then

\[
x \cdot v_k = \left( \sum_{j=1}^{n} \alpha_j v_j \right) \cdot v_k
\]

\[
= \sum_{j=1}^{n} \alpha_j (v_j \cdot v_k)
\]

\[
= \sum_{j=1}^{n} \alpha_j \delta_{jk} = \begin{cases} \alpha_k & \text{if } j = k \\ 0 & \text{if } j \neq k \end{cases}
\]

so

\[
x = \sum_{j=1}^{n} (x \cdot v_j) v_j
\]
Orthonormal Bases

**Example:** The standard basis of $\mathbb{R}^n$ is orthonormal.

$$e_i = (0, \ldots, 1, 0, \ldots, 0) \quad i = 1, \ldots, n$$

(Why?)

- **$\mathbb{R}^2$:** $e_1 = (1, 0)$, $e_2 = (0, 1)$
- **Others:** $v_1 = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$, $v_2 = \left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$

Also, many bases that are not orthonormal.
Unitary Matrices

Recall that for a real $n \times m$ matrix $A$, $A^\top$ denotes the transpose of $A$: the $(i, j)^{th}$ entry of $A^\top$ is the $(j, i)^{th}$ entry of $A$.

So the $i^{th}$ row of $A^\top$ is the $i^{th}$ column of $A$.

**Definition 2.** A real $n \times n$ matrix $A$ is unitary if $A^\top = A^{-1}$.

Notice that by definition every unitary matrix is invertible.
Unitary Matrices

**Theorem 1.** A real $n \times n$ matrix $A$ is unitary if and only if the columns of $A$ are orthonormal.

**Proof.** Let $v_j$ denote the $j^{th}$ column of $A$.

$$A^\top = A^{-1} \iff A^\top A = I = \left( \delta_{ij} \right)_{i,j=1}^n \iff v_i \cdot v_j = \delta_{ij} \quad \forall i, j \iff \{v_1, \ldots, v_n\} \text{ is orthonormal}$$
If $A$ is unitary, let $V$ be the set of columns of $A$ and $W$ be the standard basis of $\mathbb{R}^n$. Since $A$ is unitary, it is invertible, so $V$ is a basis of $\mathbb{R}^n$. (\{v_1, \ldots, v_n\} linearly independent)

$$A^\top = A^{-1} = Mtx_{V,W}(id) = \text{change of basis from } W \text{ to } V$$

Since $V$ is orthonormal, the transformation between bases $W$ and $V$ preserves all geometry, including lengths and angles.
Thus: Let $C$ be a non real symmetric matrix. Then $C$ is diagonalizable. In addition,

$$C = P^{-1}DP$$

where $D$ is a diagonal matrix and $P$ is unitary.

Note: The diagonal elements $\lambda_1, \ldots, \lambda_n$ of $D$ are the eigenvalues of $C$.

- $C$ has orthonormal eigenvectors $v_1, \ldots, v_n$ that are a basis for $\mathbb{R}^n$. 
Diagonalization of Real Symmetric Matrices

**Theorem 2.** Let $T \in L(\mathbb{R}^n, \mathbb{R}^n)$ and $W$ be the standard basis of $\mathbb{R}^n$. Suppose that $Mtx_W(T)$ is symmetric. Then the eigenvectors of $T$ are all real, and there is an orthonormal basis $V = \{v_1, \ldots, v_n\}$ of $\mathbb{R}^n$ consisting of eigenvectors of $T$, so that $Mtx_W(T)$ is diagonalizable:

$$C = Mtx_W(T) = Mtx_{W,V}(id) \cdot Mtx_V(T) \cdot Mtx_{V,W}(id)$$

where $Mtx_V T$ is diagonal and the change of basis matrices $Mtx_{V,W}(id)$ and $Mtx_{W,V}(id)$ are unitary.

The proof of the theorem requires a lengthy digression into the linear algebra of complex vector spaces. A brief outline is in the notes.
Quadratic Forms

Example: Let $f : \mathbb{R}^2 \to \mathbb{R}$

$$f(x) = \alpha x_1^2 + \beta x_1 x_2 + \gamma x_2^2$$

Let write as $f(x) = x^T A x$, A symmetric

$$A = \begin{pmatrix} \alpha & \beta \\ \beta & \gamma \end{pmatrix}$$

$$x^T A x = (x_1, x_2) \begin{pmatrix} \alpha & \beta \\ \beta & \gamma \end{pmatrix} (x_1, x_2)$$
so \( A \) is symmetric and

\[
x^\top Ax = (x_1, x_2) \begin{pmatrix}
\alpha & \frac{\beta}{2} \\
\frac{\beta}{2} & \gamma
\end{pmatrix}
\begin{pmatrix}
x_1 \\
x_2
\end{pmatrix}
\]

\[
= (x_1, x_2) \begin{pmatrix}
\alpha x_1 + \frac{\beta}{2} x_2 \\
\frac{\beta}{2} x_1 + \gamma x_2
\end{pmatrix}
\]

\[
= \alpha x_1^2 + \beta x_1 x_2 + \gamma x_2^2
\]

\[
= f(x)
\]

Notice \( f(0) = 0 \).

Can we determine anything about \( f(x) \) for \( x \neq 0 \)?

E.g., \( f(x) \geq 0 \) \( \forall x \)?

Easy if \( \beta = 0 \) ...
Quadratic Forms

**general form:**

Consider a quadratic form

\[ f(x_1, \ldots, x_n) = \sum_{i=1}^{n} \alpha_{ii} x_i^2 + \sum_{i<j} \beta_{ij} x_i x_j \]  \hspace{1cm} (1)

Let

\[ \alpha_{ij} = \begin{cases} \frac{\beta_{ij}}{2} & \text{if } i < j \\ \frac{\beta_{ji}}{2} & \text{if } i > j \end{cases} \]

Let

\[ A = \begin{pmatrix} \alpha_{11} & \cdots & \alpha_{1n} \\ \vdots & \ddots & \vdots \\ \alpha_{n1} & \cdots & \alpha_{nn} \end{pmatrix} \]

so

\[ f(x) = x^\top A x \]

\[ \text{real symmetric} \]
Quadratic Forms

$A$ is symmetric, so let $V = \{v_1, \ldots, v_n\}$ be an orthonormal basis of eigenvectors of $A$ with corresponding eigenvalues $\lambda_1, \ldots, \lambda_n$.

Then $A = U^\top DU = U\cdot D\cdot U^\top$

where $D = \begin{pmatrix}
\lambda_1 & 0 & \cdots & 0 \\
0 & \lambda_2 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & \lambda_n
\end{pmatrix}$

and $U = Mtx_{V,W}(id)$ is unitary

The columns of $U^\top$ (the rows of $U$) are the coordinates of $v_1, \ldots, v_n$, expressed in terms of the standard basis $W$. Given $x \in \mathbb{R}^n$, recall

$$x = \sum_{i=1}^n \gamma_i v_i \text{ where } \gamma_i = x \cdot v_i$$
Quadratic Forms

So

\[ \mathbf{u}^T \mathbf{A} \mathbf{x} = f(x) = f \left( \sum \gamma_i \mathbf{v}_i \right) \]

\[ = \left( \sum \gamma_i \mathbf{v}_i \right)^T A \left( \sum \gamma_i \mathbf{v}_i \right) = \mathbf{x}^T \mathbf{A} \mathbf{x} \]

\[ = \left( \sum \gamma_i \mathbf{v}_i \right)^T U^T D U \left( \sum \gamma_i \mathbf{v}_i \right) \]

\[ = \left( U \sum \gamma_i \mathbf{v}_i \right)^T D \left( U \sum \gamma_i \mathbf{v}_i \right) \]

\[ = \left( \sum \gamma_i \mathbf{U} \mathbf{v}_i \right)^T D \left( \sum \gamma_i \mathbf{U} \mathbf{v}_i \right) \]

\[ = (\gamma_1, \ldots, \gamma_n) D \begin{pmatrix} \gamma_1 \\ \vdots \\ \gamma_n \end{pmatrix} \]

\[ = \sum \lambda_i \gamma_i^2 \]

\[ \text{eigenvalues of } \mathbf{A} \]

\[ \mathbf{u} \mathbf{v}_i = e_i \quad \mathbf{x}^i \]

\[ \mathbf{u} \mathbf{v}_i = e_i \quad \mathbf{x}^i \]

\[ e_i \]
Quadratic Forms

The equation for a level set of $f$ is

$$\mathbb{R}^n : f(x) = C = \left\{ \gamma \in \mathbb{R}^n : \sum_{i=1}^{n} \lambda_i \gamma_i^2 = C \right\}$$

- If $\lambda_i \geq 0$ for all $i$, the level set is an ellipsoid, with principal axes in the directions $v_1, \ldots, v_n$. The length of the principal axis along $v_i$ is $\sqrt{C/\lambda_i}$ if $C \geq 0$ (if $\lambda_i = 0$, the level set is a degenerate ellipsoid with principal axis of infinite length in that direction). The level set is empty if $C < 0$.

  $$\implies f \text{ has global min at } 0, \ f(x) \geq 0 \ \forall x$$

- If $\lambda_i \leq 0$ for all $i$, the level set is an ellipsoid, with principal axes in the directions $v_1, \ldots, v_n$. The length of the principal

  $$\implies f \text{ has global max at } 0, \ f(x) \leq 0 \ \forall x$$
axis along $v_i$ is $\sqrt{\frac{C}{\lambda_i}}$ if $C \leq 0$ (if $\lambda_i = 0$, the level set is a degenerate ellipsoid with principal axis of infinite length in that direction). The level set is empty if $C > 0$.

• If $\lambda_i > 0$ for some $i$ and $\lambda_j < 0$ for some $j$, the level set is a hyperboloid. For example, suppose $n = 2$, $\lambda_1 > 0$, $\lambda_2 < 0$. The equation is

\[
C = \lambda_1 \gamma_1^2 + \lambda_2 \gamma_2^2 \\
= \left(\sqrt{\lambda_1 \gamma_1} + \sqrt{|\lambda_2| \gamma_2}\right) \left(\sqrt{\lambda_1 \gamma_1} - \sqrt{|\lambda_2| \gamma_2}\right)
\]

$\Rightarrow f$ has a saddle point at $0$ min with respect to $v_i$ max with respect to $v_j$
This is a hyperbola with asymptotes

\[ 0 = \sqrt{\lambda_1 \gamma_1} + \sqrt{|\lambda_2| \gamma_2} \]

\[ \Rightarrow \gamma_1 = -\sqrt{\frac{|\lambda_2|}{\lambda_1}} \gamma_2 \]

\[ 0 = \left( \sqrt{\lambda_1 \gamma_1} - \sqrt{|\lambda_2| \gamma_2} \right) \]

\[ \Rightarrow \gamma_1 = \sqrt{\frac{|\lambda_2|}{\lambda_1}} \gamma_2 \]
\( \lambda_1 > 0, \lambda_2 > 0 \)

\( \mathbf{v}_1 \)

\( \mathbf{v}_2 \)

\( \sqrt{c/\lambda_2} \)

\( \sqrt{c/\lambda_1} \)

\( \{x \in \mathbb{R}^n : f(x) = c \} \)

\( f \) has a global min at \( \mathbf{0} \)
\[ \lambda_1 > 0, \lambda_2 < 0 \]

\[ \gamma_1 = \sqrt{|\lambda_2|/\lambda_1} \quad \gamma_2 \]

\[ \gamma_1 = -\sqrt{|\lambda_2|/\lambda_1} \quad \gamma_2 \]

\[ \{x \in \mathbb{R}^n : f(x) = c\} \]

\[ f \text{ has a saddle point at } 0 \]
Quadratic Forms

This proves the following corollary of Theorem 2.

**Corollary 1.** Consider the quadratic form (1). Let \( \{v_1, \ldots, v_n\} \) be an orthonormal basis of eigenvectors of \( A \) with corresponding eigenvalues \( \{\lambda_1, \ldots, \lambda_n\} \)

1. \( f \) has a global minimum at 0 if and only if \( \lambda_i \geq 0 \) for all \( i \); the level sets of \( f \) are ellipsoids with principal axes aligned with the orthonormal eigenvectors \( v_1, \ldots, v_n \).

2. \( f \) has a global maximum at 0 if and only if \( \lambda_i \leq 0 \) for all \( i \); the level sets of \( f \) are ellipsoids with principal axes aligned with the orthonormal eigenvectors \( v_1, \ldots, v_n \).
3. If \( \lambda_i < 0 \) for some \( i \) and \( \lambda_j > 0 \) for some \( j \), then \( f \) has a saddle
point at 0; the level sets of \( f \) are hyperboloids with principal
axes aligned with the orthonormal eigenvectors \( v_1, \ldots, v_n \).
Bounded Linear Maps

**Definition 3.** Suppose $X, Y$ are normed vector spaces and $T \in L(X, Y)$. We say $T$ is bounded if

$$\exists \beta \in \mathbb{R} \text{ s.t. } \|T(x)\|_Y \leq \beta \|x\|_X \quad \forall x \in X$$

Note this implies that $T$ is Lipschitz with constant $\beta$.

why not previous notion of bounded:

$f \in \mathbb{R} \text{ s.t. } \|T(x)\| \leq B \forall x$ ?

$$T(ax) = aT(x) \quad \forall a \in \mathbb{R}$$

$$\implies \|T(ax)\| = |a| \|T(x)\| \quad \forall a \in \mathbb{R}$$
Bounded Linear Maps

Much more is true:

**Theorem 3** (Thms. 4.1, 4.3). Let $X$ and $Y$ be normed vector spaces and $T \in L(X,Y)$. Then

$$T \text{ is continuous at some point } x_0 \in X \iff T \text{ is continuous at every } x \in X$$

$$\iff T \text{ is uniformly continuous on } X$$

$$\iff T \text{ is Lipschitz}$$

$$\iff T \text{ is bounded}$$

**Proof.** Suppose $T$ is continuous at $x_0$. Fix $\varepsilon > 0$. Then there exists $\delta > 0$ such that

$$\|z - x_0\| < \delta \Rightarrow \|T(z) - T(x_0)\| < \varepsilon$$
Now suppose \( x \) is any element of \( X \). If \( \|y - x\| < \delta \), let \( z = y - x + x_0 \), so \( \|z - x_0\| = \|y - x\| < \delta \).

\[
\begin{align*}
\|T(y) - T(x)\| & = \|T(y - x)\| \quad \text{(T linear)} \\
& = \|T(y - x + x_0 - x_0)\| \quad = \|T(z - x_0)\| \\
& = \|T(z) - T(x_0)\| \quad \text{(T linear)} \\
& < \varepsilon
\end{align*}
\]

which proves that \( T \) is continuous at every \( x \), and uniformly continuous.

We claim that \( T \) is bounded if and only if \( T \) is continuous at 0. Suppose \( T \) is not bounded. Then

\[
\exists \{x_n\} \text{ s.t. } \|T(x_n)\| > n\|x_n\| \quad \forall n
\]
Note that $x_n \neq 0$. Let $\varepsilon = 1$. Fix $\delta > 0$ and choose $n$ such that $\frac{1}{n} < \delta$. Let

$$x'_n = \frac{x_n}{n\|x_n\|} = \frac{1}{n} \frac{x_n}{\|x_n\|}$$

so

$$\|x'_n\| = \frac{\|x_n\|}{n\|x_n\|} = \frac{1}{n} \frac{\|x_n\|}{\|x_n\|} = \frac{1}{n}$$

$$\|x'_n - 0\| < \frac{\varepsilon}{n} \delta$$

$$\|T(x'_n) - T(0)\| = \|T(x'_n)\| = \frac{1}{n\|x_n\|} \|T(x_n)\|$$

\[
\begin{align*}
\quad & < \\
\quad & = 1 \\
\quad & = \varepsilon
\end{align*}
\]
Since this is true for every \( \delta \), \( T \) is not continuous at 0. Therefore, \( T \) continuous at 0 implies \( T \) is bounded. Now, suppose \( T \) is bounded, so find \( M \) such that \( \|T(x)\| \leq M\|x\| \) for every \( x \in X \). Given \( \varepsilon > 0 \), let \( \delta = \varepsilon/M \). Then

\[
\|x - 0\| < \delta \implies \|x\| < \delta
\]

\[
\implies \|T(x) - T(0)\| = \|T(x)\| < M\delta
\]

\[
\implies \|T(x) - T(0)\| < \varepsilon = M\delta
\]

so \( T \) is continuous at 0.

Thus, we have shown that continuity at some point \( x_0 \) implies uniform continuity, which implies continuity at every point, which implies \( T \) is continuous at 0, which implies that \( T \) is bounded, which implies that \( T \) is continuous at 0, which implies that \( T \) is
continuous at some $x_0$, so all of the statements except possibly the Lipschitz statement are equivalent.

Suppose $T$ is bounded, with constant $M > 0$. Then
\[
\|T(x) - T(y)\| = \|T(x - y)\| \leq M\|x - y\|
\]
so $T$ is Lipschitz with constant $M$; conversely, if $T$ is Lipschitz with constant $M$, then $T$ is bounded with constant $M$. So all the statements are equivalent. □

\[ \forall x \in X : \|T(x) - T(0)\| = \|T(x)\| \leq M\|x - 0\| = M\|x\|. \]
Bounded Linear Maps

Every linear map on a finite-dimensional normed vector space is bounded (and thus continuous, uniformly continuous, and Lipschitz continuous).

**Theorem 4** (Thm. 4.5). Let $X$ and $Y$ be normed vector spaces, with $\dim X = n$. Every $T \in L(X, Y)$ is bounded.

\[
\left( \forall n \in \mathbb{N} \right)
\]

**Proof.** See de la Fuente. \qed
Topological Isomorphism

Definition 4. A topological isomorphism between normed vector spaces $X$ and $Y$ is a linear transformation $T \in L(X,Y)$ that is invertible (one-to-one, onto), continuous, and has a continuous inverse.

Two normed vector spaces $X$ and $Y$ are topologically isomorphic if there is a topological isomorphism $T : X \to Y$. 
The Space $B(X,Y)$

Suppose $X$ and $Y$ are normed vector spaces. We define

$$B(X,Y) = \{ T \in L(X,Y) : T \text{ is bounded} \}$$

Define:

$$\|T\|_{B(X,Y)} = \sup \left\{ \frac{\|T(x)\|_Y}{\|x\|_X} : x \in X, x \neq 0 \right\}$$

$$= \sup \{ \|T(x)\|_Y : \|x\|_X = 1 \}$$

$$\Rightarrow \quad \|T(x)\|_Y \leq \|T\| \|x\|_X \quad \forall x \in X \quad \text{by definition}$$

We skip the proofs of the rest of these results – read dIF.
The Space $B(X, Y)$

**Theorem 5** (Thm. 4.8). Let $X, Y$ be normed vector spaces. Then

$$
\left( B(X, Y), \| \cdot \|_{B(X,Y)} \right)
$$

is a normed vector space.
The Space $B(\mathbb{R}^n, \mathbb{R}^m)$

**Theorem 6** (Thm. 4.9). Let $T \in L(\mathbb{R}^n, \mathbb{R}^m) (= B(\mathbb{R}^n, \mathbb{R}^m))$ with matrix $A = (a_{ij})$ with respect to the standard bases. Let

$$M = \max\{|a_{ij}| : 1 \leq i \leq m, 1 \leq j \leq n\}$$

Then

$$M \leq \|T\| \leq M\sqrt{mn}.$$
Compositions

**Theorem 7** (Thm. 4.10). Let $R \in L(\mathbb{R}^m, \mathbb{R}^n)$ and $S \in L(\mathbb{R}^n, \mathbb{R}^p)$. Then

$$\|S \circ R\| \leq \|S\| \|R\|$$
Invertibility

Define $\Omega(\mathbb{R}^n) = \{T \in L(\mathbb{R}^n, \mathbb{R}^n) : T \text{ is invertible}\}$

**Theorem 8 (Thm. 4.11').** Suppose $T \in L(\mathbb{R}^n, \mathbb{R}^n)$ and $E$ is the standard basis of $\mathbb{R}^n$. Then

$T$ is invertible

$\iff$ ker $T = \{0\}$

$\iff$ det $(\text{Mat}_E(T)) \neq 0$

$\iff$ det $(\text{Mat}_V(T)) \neq 0$ for every basis $V$

$\iff$ det $(\text{Mat}_{V,W}(T)) \neq 0$ for every pair of bases $V, W$
Invertibility

**Theorem 9 (Thm. 4.12).** If $S, T \in \Omega(\mathbb{R}^n)$, then $S \circ T \in \Omega(\mathbb{R}^n)$ and

$$(S \circ T)^{-1} = T^{-1} \circ S^{-1}$$
Invertibility

**Theorem 10** (Thm. 4.14). Let $S, T \in L(\mathbb{R}^n, \mathbb{R}^n)$. If $T$ is invertible and

$$\|T - S\| < \frac{1}{\|T^{-1}\|}$$

then $S$ is invertible. In particular, $\Omega(\mathbb{R}^n)$ is open in $L(\mathbb{R}^n, \mathbb{R}^n) = B(\mathbb{R}^n, \mathbb{R}^n)$.

**Theorem 11** (Thm. 4.15). The function $(\cdot)^{-1} : \Omega(\mathbb{R}^n) \to \Omega(\mathbb{R}^n)$ that assigns $T^{-1}$ to each $T \in \Omega(\mathbb{R}^n)$ is continuous.