Econ 204 2019

Lecture 10

Outline

1. Diagonalization of Real Symmetric Matrices
2. Application to Quadratic Forms
3. Linear Maps Between Normed Spaces
How Might This Matter

- Why does diagonalizability matter?

Consider a two-dimensional linear difference equation:

\[
\begin{pmatrix}
c_{t+1} \\
k_{t+1}
\end{pmatrix} =
\begin{pmatrix}
b_{11} & b_{12} \\
b_{21} & b_{22}
\end{pmatrix}
\begin{pmatrix}
c_t \\
k_t
\end{pmatrix} \quad \forall t = 0, 1, 2, 3, \ldots
\]

given an initial condition \( c_0, k_0 \), or, setting

\[
y_t = \begin{pmatrix}
c_t \\
k_t
\end{pmatrix} \quad \forall t \quad \text{and} \quad B = \begin{pmatrix}
b_{11} & b_{12} \\
b_{21} & b_{22}
\end{pmatrix}
\]

we can rewrite this more compactly as

\[
y_{t+1} = By_t \quad \forall t
\]

where \( b_{ij} \in \mathbb{R} \) each \( i, j \).
We want to find a solution $y_t$, $t = 1, 2, 3, \ldots$ given initial condition $y_0$. (Why?)

Such a dynamical system will arise for example as a characterization of the solution to a standard infinite-horizon optimal growth problem (202a, lecture 2).

If $B$ is diagonalizable, this can be easily solved after a change of basis. If $B$ is diagonalizable, choose an invertible $2 \times 2$ real matrix $P$ such that

$$P^{-1}BP = D = \begin{pmatrix} d_1 & 0 \\ 0 & d_2 \end{pmatrix}$$

Then

$$y_{t+1} = By_t \quad \forall t \iff P^{-1}y_{t+1} = P^{-1}By_t \quad \forall t$$
$$\iff P^{-1}y_{t+1} = P^{-1}BPP^{-1}y_t \quad \forall t$$
$$\iff \bar{y}_{t+1} = D\bar{y}_t \quad \forall t$$
where $\bar{y}_t = P^{-1}y_t \ \forall t$.

Since $D$ is diagonal, after a change of basis to $\bar{y}_t$, we need to solve two independent linear univariate difference equations, which is easy:

$$\bar{y}_{it} = d_i^t\bar{y}_{i0} \ \forall t$$

- Not all real $n \times n$ matrices are diagonalizable (not even all invertible $n \times n$ matrices are)...so can we identify some classes that are?

- Some types of matrices appear more frequently than others – especially real symmetric $n \times n$ matrices (matrix representation of second derivatives of $C^2$ functions, quadratic forms...).
• Recall that an $n \times n$ real matrix $A$ is symmetric if $a_{ij} = a_{ji}$ for all $i, j$, where $a_{ij}$ is the $(i,j)^{th}$ entry of $A$. 
Orthonormal Bases

Definition 1. Let

\[ \delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases} \]

A basis \( V = \{v_1, \ldots, v_n\} \) of \( \mathbb{R}^n \) is orthonormal if \( v_i \cdot v_j = \delta_{ij} \).

In other words, a basis is orthonormal if each basis element has unit length (\( \|v_i\|^2 = v_i \cdot v_i = 1 \ \forall i \)), and distinct basis elements are perpendicular (\( v_i \cdot v_j = 0 \) for \( i \neq j \)).
Orthonormal Bases

Remark: Suppose that \( x = \sum_{j=1}^{n} \alpha_j v_j \) where \( \{v_1, \ldots, v_n\} \) is an orthonormal basis of \( \mathbb{R}^n \). Then

\[
x \cdot v_k = \left( \sum_{j=1}^{n} \alpha_j v_j \right) \cdot v_k
\]

\[
= \sum_{j=1}^{n} \alpha_j (v_j \cdot v_k)
\]

\[
= \sum_{j=1}^{n} \alpha_j \delta_{jk}
\]

\[
= \alpha_k
\]

so

\[
x = \sum_{j=1}^{n} (x \cdot v_j) v_j
\]
Orthonormal Bases

**Example:** The standard basis of $\mathbb{R}^n$ is orthonormal.

(Why?)
Unitary Matrices

Recall that for a real $n \times m$ matrix $A$, $A^\top$ denotes the transpose of $A$: the $(i, j)^{th}$ entry of $A^\top$ is the $(j, i)^{th}$ entry of $A$.

So the $i^{th}$ row of $A^\top$ is the $i^{th}$ column of $A$.

**Definition 2.** A real $n \times n$ matrix $A$ is unitary if $A^\top = A^{-1}$.

Notice that by definition every unitary matrix is invertible.
Unitary Matrices

**Theorem 1.** A real $n \times n$ matrix $A$ is unitary if and only if the columns of $A$ are orthonormal.

**Proof.** Let $v_j$ denote the $j^{th}$ column of $A$.

$$A^\top = A^{-1} \iff A^\top A = I$$
$$\iff v_i \cdot v_j = \delta_{ij} \quad \forall i, j$$
$$\iff \{v_1, \ldots, v_n\} \text{ is orthonormal}$$
Unitary Matrices

If $A$ is unitary, let $V$ be the set of columns of $A$ and $W$ be the standard basis of $\mathbb{R}^n$. Since $A$ is unitary, it is invertible, so $V$ is a basis of $\mathbb{R}^n$.

$$A^\top = A^{-1} = Mtx_{V,W}(id)$$

Since $V$ is orthonormal, the transformation between bases $W$ and $V$ preserves all geometry, including lengths and angles.
Diagonalization of Real Symmetric Matrices

**Theorem 2.** Let $T \in L(\mathbb{R}^n, \mathbb{R}^n)$ and $W$ be the standard basis of $\mathbb{R}^n$. Suppose that $Mtx_W(T)$ is symmetric. Then the eigenvectors of $T$ are all real, and there is an orthonormal basis $V = \{v_1, \ldots, v_n\}$ of $\mathbb{R}^n$ consisting of eigenvectors of $T$, so that $Mtx_W(T)$ is diagonalizable:

$$Mtx_W(T) = Mtx_{W,V}(id) \cdot Mtx_V(T) \cdot Mtx_{V,W}(id)$$

where $Mtx_V T$ is diagonal and the change of basis matrices $Mtx_{V,W}(id)$ and $Mtx_{W,V}(id)$ are unitary.

The proof of the theorem requires a lengthy digression into the linear algebra of complex vector spaces. A brief outline is in the notes.
Quadratic Forms

Example: Let

\[ f(x) = \alpha x_1^2 + \beta x_1 x_2 + \gamma x_2^2 \]

Let

\[
A = \begin{pmatrix} \alpha & \frac{\beta}{2} \\ \frac{\beta}{2} & \gamma \end{pmatrix}
\]
so $A$ is symmetric and

$$x^\top Ax = (x_1, x_2) \begin{pmatrix} \alpha & \frac{\beta}{2} \\ \frac{\beta}{2} & \gamma \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$= (x_1, x_2) \begin{pmatrix} \alpha x_1 + \frac{\beta}{2} x_2 \\ \frac{\beta}{2} x_1 + \gamma x_2 \end{pmatrix}$$

$$= \alpha x_1^2 + \beta x_1 x_2 + \gamma x_2^2$$

$$= f(x)$$
Quadratic Forms

Consider a quadratic form

\[ f(x_1, \ldots, x_n) = \sum_{i=1}^{n} \alpha_{ii} x_i^2 + \sum_{i<j} \beta_{ij} x_i x_j \]  \hspace{1cm} (1)

Let

\[ \alpha_{ij} = \begin{cases} \frac{\beta_{ij}}{2} & \text{if } i < j \\ \frac{\beta_{ii}}{2} & \text{if } i > j \end{cases} \]

Let

\[ A = \begin{pmatrix} \alpha_{11} & \cdots & \alpha_{1n} \\ \vdots & \ddots & \vdots \\ \alpha_{n1} & \cdots & \alpha_{nn} \end{pmatrix} \]  \hspace{1cm} \text{so } f(x) = x^T A x \]
Quadratic Forms

$A$ is symmetric, so let $V = \{v_1, \ldots, v_n\}$ be an orthonormal basis of eigenvectors of $A$ with corresponding eigenvalues $\lambda_1, \ldots, \lambda_n$.

Then $A = U^\top D U$

where $D = \begin{pmatrix} 
\lambda_1 & 0 & \cdots & 0 \\
0 & \lambda_2 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & \lambda_n 
\end{pmatrix}$

and $U = Mtx_{V,W}(id)$ is unitary

The columns of $U^\top$ (the rows of $U$) are the coordinates of $v_1, \ldots, v_n$, expressed in terms of the standard basis $W$. Given $x \in \mathbb{R}^n$, recall

$$x = \sum_{i=1}^{n} \gamma_i v_i \text{ where } \gamma_i = x \cdot v_i$$
Quadratic Forms

So

\[ f(x) = f \left( \sum \gamma_i v_i \right) \]
\[ = \left( \sum \gamma_i v_i \right)^T A \left( \sum \gamma_i v_i \right) \]
\[ = \left( \sum \gamma_i v_i \right)^T U^T D U \left( \sum \gamma_i v_i \right) \]
\[ = \left( U \sum \gamma_i v_i \right)^T D \left( U \sum \gamma_i v_i \right) \]
\[ = \left( \sum \gamma_i U v_i \right)^T D \left( \sum \gamma_i U v_i \right) \]
\[ = \left( \gamma_1, \ldots, \gamma_n \right) D \begin{pmatrix} \gamma_1 \\ \vdots \\ \gamma_n \end{pmatrix} \]
\[ = \sum \lambda_i \gamma_i^2 \]
Quadratic Forms

The equation for a level set of $f$ is

$$
\left\{ \gamma \in \mathbb{R}^n : \sum_{i=1}^{n} \lambda_i \gamma_i^2 = C \right\}
$$

- If $\lambda_i \geq 0$ for all $i$, the level set is an ellipsoid, with principal axes in the directions $v_1, \ldots, v_n$. The length of the principal axis along $v_i$ is $\sqrt{C/\lambda_i}$ if $C \geq 0$ (if $\lambda_i = 0$, the level set is a degenerate ellipsoid with principal axis of infinite length in that direction). The level set is empty if $C < 0$.

- If $\lambda_i \leq 0$ for all $i$, the level set is an ellipsoid, with principal axes in the directions $v_1, \ldots, v_n$. The length of the principal
axis along $v_i$ is $\sqrt{C/\lambda_i}$ if $C \leq 0$ (if $\lambda_i = 0$, the level set is a degenerate ellipsoid with principal axis of infinite length in that direction). The level set is empty if $C > 0$.

- If $\lambda_i > 0$ for some $i$ and $\lambda_j < 0$ for some $j$, the level set is a hyperboloid. For example, suppose $n = 2$, $\lambda_1 > 0$, $\lambda_2 < 0$. The equation is

\[
C = \lambda_1 \gamma_1^2 + \lambda_2 \gamma_2^2 \\
= \left(\sqrt{\lambda_1 \gamma_1} + \sqrt{|\lambda_2| \gamma_2}\right) \left(\sqrt{\lambda_1 \gamma_1} - \sqrt{|\lambda_2| \gamma_2}\right)
\]
This is a hyperbola with asymptotes

\[0 = \sqrt{\lambda_1 \gamma_1} + \sqrt{|\lambda_2| \gamma_2}\]

\[\Rightarrow \gamma_1 = -\sqrt{\frac{|\lambda_2|}{\lambda_1}} \gamma_2\]

\[0 = \left(\sqrt{\lambda_1 \gamma_1} - \sqrt{|\lambda_2| \gamma_2}\right)\]

\[\Rightarrow \gamma_1 = \sqrt{\frac{|\lambda_2|}{\lambda_1}} \gamma_2\]
\[ \lambda_1 > 0, \lambda_2 > 0 \]
$\lambda_1 > 0, \lambda_2 < 0$

$\nu_1 = \nu |\lambda_2|/\lambda_1$
Quadratic Forms

This proves the following corollary of Theorem 2.

**Corollary 1.** Consider the quadratic form (1).

1. $f$ has a global minimum at 0 if and only if $\lambda_i \geq 0$ for all $i$; the level sets of $f$ are ellipsoids with principal axes aligned with the orthonormal eigenvectors $v_1, \ldots, v_n$.

2. $f$ has a global maximum at 0 if and only if $\lambda_i \leq 0$ for all $i$; the level sets of $f$ are ellipsoids with principal axes aligned with the orthonormal eigenvectors $v_1, \ldots, v_n$. 
3. If $\lambda_i < 0$ for some $i$ and $\lambda_j > 0$ for some $j$, then $f$ has a saddle point at 0; the level sets of $f$ are hyperboloids with principal axes aligned with the orthonormal eigenvectors $v_1, \ldots, v_n$. 
Bounded Linear Maps

Definition 3. Suppose \( X, Y \) are normed vector spaces and \( T \in L(X, Y) \). We say \( T \) is bounded if

\[
\exists \beta \in \mathbb{R} \text{ s.t. } \|T(x)\|_Y \leq \beta \|x\|_X \quad \forall x \in X
\]

Note this implies that \( T \) is Lipschitz with constant \( \beta \).
Bounded Linear Maps

Much more is true:

**Theorem 3** (Thms. 4.1, 4.3). Let $X$ and $Y$ be normed vector spaces and $T \in L(X, Y)$. Then

- $T$ is continuous at some point $x_0 \in X$ if and only if $T$ is continuous at every $x \in X$.
- $T$ is uniformly continuous on $X$ if and only if $T$ is Lipschitz.
- $T$ is Lipschitz if and only if $T$ is bounded.

**Proof.** Suppose $T$ is continuous at $x_0$. Fix $\varepsilon > 0$. Then there exists $\delta > 0$ such that

$$\|z - x_0\| < \delta \implies \|T(z) - T(x_0)\| < \varepsilon$$
Now suppose $x$ is any element of $X$. If $\|y - x\| < \delta$, let $z = y - x + x_0$, so $\|z - x_0\| = \|y - x\| < \delta$.

\[
||T(y) - T(x)||
= ||T(y - x)||
= ||T(y - x + x_0 - x_0)||
= ||T(z) - T(x_0)||
< \varepsilon
\]

which proves that $T$ is continuous at every $x$, and uniformly continuous.

We claim that $T$ is bounded if and only if $T$ is continuous at 0. Suppose $T$ is not bounded. Then

\[
\exists\{x_n\} \text{ s.t. } ||T(x_n)|| > n\|x_n\| \quad \forall n
\]
Note that \( x_n \neq 0 \). Let \( \varepsilon = 1 \). Fix \( \delta > 0 \) and choose \( n \) such that \( \frac{1}{n} < \delta \). Let

\[
\begin{align*}
x'_n &= \frac{x_n}{n\|x_n\|} \\
\|x'_n\| &= \frac{\|x_n\|}{n\|x_n\|} \\
&= \frac{1}{n} < \delta \\
\|T(x'_n) - T(0)\| &= \|T(x'_n)\| \\
&= \frac{1}{n\|x_n\|}\|T(x_n)\| \\
&> \frac{n\|x_n\|}{n\|x_n\|} \\
&= 1 \\
&= \varepsilon
\end{align*}
\]
Since this is true for every $\delta$, $T$ is not continuous at 0. Therefore, $T$ continuous at 0 implies $T$ is bounded. Now, suppose $T$ is bounded, so find $M$ such that $\|T(x)\| \leq M\|x\|$ for every $x \in X$. Given $\varepsilon > 0$, let $\delta = \varepsilon/M$. Then

$$\|x - 0\| < \delta \Rightarrow \|x\| < \delta$$

$$\Rightarrow \|T(x) - T(0)\| = \|T(x)\| < M\delta$$

$$\Rightarrow \|T(x) - T(0)\| < \varepsilon$$

so $T$ is continuous at 0.

Thus, we have shown that continuity at some point $x_0$ implies uniform continuity, which implies continuity at every point, which implies $T$ is continuous at 0, which implies that $T$ is bounded, which implies that $T$ is continuous at 0, which implies that $T$ is
continuous at some $x_0$, so all of the statements except possibly the Lipschitz statement are equivalent.

Suppose $T$ is bounded, with constant $M$. Then

$$
\|T(x) - T(y)\| = \|T(x - y)\| 
\leq M\|x - y\|
$$

so $T$ is Lipschitz with constant $M$; conversely, if $T$ is Lipschitz with constant $M$, then $T$ is bounded with constant $M$. So all the statements are equivalent. $\square$
Bounded Linear Maps

Every linear map on a finite-dimensional normed vector space is bounded (and thus continuous, uniformly continuous, and Lipschitz continuous).

**Theorem 4** (Thm. 4.5). Let $X$ and $Y$ be normed vector spaces, with $\dim X = n$. Every $T \in L(X, Y)$ is bounded.

*Proof.* See de la Fuente. □
Topological Isomorphism

Definition 4. A topological isomorphism between normed vector spaces $X$ and $Y$ is a linear transformation $T \in L(X,Y)$ that is invertible (one-to-one, onto), continuous, and has a continuous inverse.

Two normed vector spaces $X$ and $Y$ are topologically isomorphic if there is a topological isomorphism $T : X \to Y$. 
The Space $B(X, Y)$

Suppose $X$ and $Y$ are normed vector spaces. We define

$$B(X, Y) = \{T \in L(X, Y) : T \text{ is bounded}\}$$

$$\|T\|_{B(X,Y)} = \sup \left\{ \frac{\|T(x)\|_Y}{\|x\|_X}, x \in X, x \neq 0 \right\}$$

$$= \sup \{\|T(x)\|_Y : \|x\|_X = 1\}$$

We skip the proofs of the rest of these results – read dLF.
The Space $B(X, Y)$

**Theorem 5 (Thm. 4.8).** Let $X, Y$ be normed vector spaces. Then

$$\left( B(X, Y), \| \cdot \|_{B(X,Y)} \right)$$

is a normed vector space.
The Space $B(\mathbb{R}^n, \mathbb{R}^m)$

**Theorem 6** (Thm. 4.9). Let $T \in L(\mathbb{R}^n, \mathbb{R}^m) (= B(\mathbb{R}^n, \mathbb{R}^m))$ with matrix $A = (a_{ij})$ with respect to the standard bases. Let

$$M = \max\{|a_{ij}| : 1 \leq i \leq m, 1 \leq j \leq n\}$$

Then

$$M \leq \|T\| \leq M\sqrt{mn}$$
Compositions

Theorem 7 (Thm. 4.10). Let $R \in L(\mathbb{R}^m, \mathbb{R}^n)$ and $S \in L(\mathbb{R}^n, \mathbb{R}^p)$. Then

$$\|S \circ R\| \leq \|S\| \|R\|$$
Invertibility

Define $\Omega(\mathbb{R}^n) = \{T \in L(\mathbb{R}^n, \mathbb{R}^n) : T \text{ is invertible}\}$

**Theorem 8** (Thm. 4.11'). Suppose $T \in L(\mathbb{R}^n, \mathbb{R}^n)$ and $E$ is the standard basis of $\mathbb{R}^n$. Then

$T$ is invertible

$\iff$ \hspace{1em} $\ker T = \{0\}$

$\iff$ \hspace{1em} $\det (\text{Matrix}_E(T)) \neq 0$

$\iff$ \hspace{1em} $\det (\text{Matrix}_{V,V}(T)) \neq 0$ for every basis $V$

$\iff$ \hspace{1em} $\det (\text{Matrix}_{V,W}(T)) \neq 0$ for every pair of bases $V, W$
Invertibility

Theorem 9 (Thm. 4.12). If \( S, T \in \Omega(\mathbb{R}^n) \), then \( S \circ T \in \Omega(\mathbb{R}^n) \) and \[
(S \circ T)^{-1} = T^{-1} \circ S^{-1}
\]
Invertibility

**Theorem 10** (Thm. 4.14). Let \( S, T \in L(\mathbb{R}^n, \mathbb{R}^n) \). If \( T \) is invertible and
\[
\|T - S\| < \frac{1}{\|T^{-1}\|}
\]
then \( S \) is invertible. In particular, \( \Omega(\mathbb{R}^n) \) is open in \( L(\mathbb{R}^n, \mathbb{R}^n) = B(\mathbb{R}^n, \mathbb{R}^n) \).

**Theorem 11** (Thm. 4.15). The function \((\cdot)^{-1} : \Omega(\mathbb{R}^n) \to \Omega(\mathbb{R}^n)\) that assigns \( T^{-1} \) to each \( T \in \Omega(\mathbb{R}^n) \) is continuous.