

## Announcements

- PS 4 due Tuesday
- comments about exam Wednesday or Thursday
- review session by Farzat + Dami Friday 5pm 648 Evans

# Econ 204 2019

## Lecture 11

### Outline

1. Derivatives
2. Chain Rule
3. Mean Value Theorem
4. Taylor's Theorem

# Derivatives

**Definition 1.** Let  $f : I \rightarrow \mathbf{R}$ , where  $I \subseteq \mathbf{R}$  is an open interval.  $f$  is differentiable at  $x \in I$  if

$$\lim_{h \rightarrow 0} \frac{f(x + h) - f(x)}{h} = a$$

for some  $a \in \mathbf{R}$ .

This is equivalent to  $\exists a \in \mathbf{R}$  such that

$$\lim_{h \rightarrow 0} \frac{f(x+h) - (f(x) + ah)}{h} = 0$$

$$\Leftrightarrow \forall \varepsilon > 0 \exists \delta > 0 \text{ s.t. } 0 < |h| < \delta \Rightarrow \left| \frac{f(x+h) - (f(x) + ah)}{h} \right| < \varepsilon$$

$$\Leftrightarrow \forall \varepsilon > 0 \exists \delta > 0 \text{ s.t. } 0 < |h| < \delta \Rightarrow \frac{|f(x+h) - (f(x) + ah)|}{|h|} < \varepsilon$$

$$\Leftrightarrow \lim_{h \rightarrow 0} \frac{|f(x+h) - (f(x) + ah)|}{|h|} = 0$$

Notice  $T: \mathbb{R} \rightarrow \mathbb{R}$  is a linear transformation  
 $\Leftrightarrow T(h) = rh$  for some  $r \in \mathbb{R}$

# Derivatives

**Definition 2.** If  $X \subseteq \mathbf{R}^n$  is open,  $f : X \rightarrow \mathbf{R}^m$  is differentiable at  $x \in X$  if  $\exists T_x \in L(\mathbf{R}^n, \mathbf{R}^m)$  such that

$$\lim_{h \rightarrow 0, h \in \mathbf{R}^n} \frac{\|f(x+h) - (f(x) + T_x(h))\|}{\|h\|} = 0 \quad (1)$$

$f$  is differentiable if it is differentiable at all  $x \in X$ .

Note that  $T_x$  is uniquely determined by Equation (1).

The definition requires that **one** linear <sup>transformation</sup> ~~operator~~  $T_x$  works no matter how  $h$  approaches zero.

In this case,  $f(x) + T_x(h)$  is the best linear approximation to  $f(x+h)$  for sufficiently small  $h$ .

# Big-Oh and little-oh

## Notation:

- $y = O(|h|^n)$  as  $h \rightarrow 0$  – read “ $y$  is big-Oh of  $|h|^n$ ” – means

$$\exists K, \delta > 0 \text{ s.t. } |h| < \delta \Rightarrow |y| \leq K|h|^n$$

$\frac{\|y\|}{\|h\|^n}$  is bounded as  $h \rightarrow 0$

- $y = o(|h|^n)$  as  $h \rightarrow 0$  – read “ $y$  is little-oh of  $|h|^n$ ” – means

$$\lim_{h \rightarrow 0} \frac{|y|}{|h|^n} = 0$$

$\frac{\|y\|}{\|h\|^n} \rightarrow 0$  as  $h \rightarrow 0$

• Nested:  $o(\|h\|^n) \Rightarrow O(\|h\|^n)$

• Note that  $y = O(\|h\|^{n+1})$  as  $h \rightarrow 0$  implies  $y = o(\|h\|^n)$  as  $h \rightarrow 0$ .

• Also  $y = O(\|h\|^n)$  or  $y = o(\|h\|^n) \Rightarrow y \rightarrow 0$  as  $h \rightarrow 0$ . 4

Using this notation:  $f$  is differentiable at  $x \Leftrightarrow \exists T_x \in L(\mathbf{R}^n, \mathbf{R}^m)$   
such that

$$f(x + h) = f(x) + T_x(h) + o(h) \text{ as } h \rightarrow 0$$

# More Notation

## Notation:

- $df_x$  is the linear transformation  $T_x$
- $Df(x)$  is the matrix <sup>representation</sup> of  $df_x$  with respect to the standard basis.  
This is called the *Jacobian* or *Jacobian matrix* of  $f$  at  $x$
- $E_f(h) = f(x + h) - (f(x) + df_x(h))$  is the *error term*

Using this notation,

$$f \text{ is differentiable at } x \Leftrightarrow E_f(h) = o(h) \text{ as } h \rightarrow 0$$

# What's $Df(x)$ ?

matrix representation of  $T_x = df_x$

Now compute  $Df(x) = (a_{ij})$ . Let  $\{e_1, \dots, e_n\}$  be the standard basis of  $\mathbf{R}^n$ . Look in direction  $e_j$  (note that  $\|\gamma e_j\| = |\gamma|$ ).

$\gamma e_j \rightarrow 0$  as  $\gamma \rightarrow 0$

$$\begin{aligned}
 o(\gamma) &= f(x + \gamma e_j) - \left( f(x) + \underbrace{T_x(\gamma e_j)}_{df_x(\gamma e_j)} \right) \\
 &= f(x + \gamma e_j) - \left( f(x) + \begin{pmatrix} a_{11} & \cdots & a_{1j} & \cdots & a_{1n} \\ \vdots & \cdots & \vdots & \cdots & \vdots \\ a_{m1} & \cdots & a_{mj} & \cdots & a_{mn} \end{pmatrix} \begin{pmatrix} 0 \\ \vdots \\ 0 \\ \gamma \\ 0 \\ \vdots \\ 0 \end{pmatrix} \right) \\
 &= f(x + \gamma e_j) - \left( f(x) + \begin{pmatrix} \gamma a_{1j} \\ \vdots \\ \gamma a_{mj} \end{pmatrix} \right)
 \end{aligned}$$

$j^{\text{th}}$  spot



$$f: \mathbb{R}^n \rightarrow \mathbb{R}^m, \quad f = (f^1, \dots, f^m)$$

For  $i = 1, \dots, m$ , let  $f^i$  denote the  $i^{\text{th}}$  component of the function

$$f: \quad f(x) = (f^1(x), \dots, f^m(x)), \quad f^i: \mathbb{R}^n \rightarrow \mathbb{R} \quad \forall i$$

$$\forall i: \quad f^i(x + \gamma e_j) - (f^i(x) + \gamma a_{ij}) = o(\gamma)$$

$$\text{so } a_{ij} = \frac{\partial f^i}{\partial x_j}(x)$$

## Derivatives and Partial Derivatives

**Theorem 1** (Thm. 3.3). *Suppose  $X \subseteq \mathbf{R}^n$  is open and  $f : X \rightarrow \mathbf{R}^m$  is differentiable at  $x \in X$ . Then  $\frac{\partial f^i}{\partial x_j}(x)$  exists for  $1 \leq i \leq m$ ,  $1 \leq j \leq n$ , and*

$$Df(x) = \begin{pmatrix} \frac{\partial f^1}{\partial x_1}(x) & \cdots & \frac{\partial f^1}{\partial x_n}(x) \\ \vdots & \cdots & \vdots \\ \frac{\partial f^m}{\partial x_1}(x) & \cdots & \frac{\partial f^m}{\partial x_n}(x) \end{pmatrix}$$

*i.e. the Jacobian at  $x$  is the matrix of partial derivatives at  $x$ .*

# Derivatives and Partial Derivatives

**Remark:** If  $f$  is differentiable at  $x$ , then all first-order partial derivatives  $\frac{\partial f^i}{\partial x_j}$  exist at  $x$ . However, the converse is false: existence of all the first-order partial derivatives does not imply that  $f$  is differentiable.

The missing piece is continuity of the partial derivatives:

**Theorem 2** (Thm. 3.4). *If all the first-order partial derivatives  $\frac{\partial f^i}{\partial x_j}$  ( $1 \leq i \leq m$ ,  $1 \leq j \leq n$ ) exist and are continuous at  $x$ , then  $f$  is differentiable at  $x$ .*

## Directional Derivatives

Suppose  $X \subseteq \mathbf{R}^n$  open,  $f : X \rightarrow \mathbf{R}^m$  is differentiable at  $x$ , and  $\|u\| = 1$ .  $u \in \mathbf{R}^n$ ,  $\gamma u \rightarrow 0$  as  $\gamma \rightarrow 0$  :  $\|\gamma u\| = |\gamma| \|u\| = |\gamma|$

$$f(x + \gamma u) - (f(x) + T_x(\gamma u)) = o(\gamma) \text{ as } \gamma \rightarrow 0$$

$$\Rightarrow f(x + \gamma u) - (f(x) + \gamma T_x(u)) = o(\gamma) \text{ as } \gamma \rightarrow 0 \quad (T_x \text{ linear})$$

$$\Rightarrow \lim_{\gamma \rightarrow 0} \frac{f(x + \gamma u) - f(x)}{\gamma} = T_x(u) = Df(x)u$$

$= df_x(u)$

i.e. the directional derivative in the direction  $u$  (with  $\|u\| = 1$ ) is

$$Df(x)u \in \mathbf{R}^m$$

# Chain Rule

**Theorem 3** (Thm. 3.5, Chain Rule). *Let  $X \subseteq \mathbf{R}^n$ ,  $Y \subseteq \mathbf{R}^m$  be open,  $f : X \rightarrow Y$ ,  $g : Y \rightarrow \mathbf{R}^p$ . Let  $x_0 \in X$  and  $F = g \circ f$ . If  $f$  is differentiable at  $x_0$  and  $g$  is differentiable at  $f(x_0)$ , then  $F = g \circ f$  is differentiable at  $x_0$  and*

$$dF_{x_0} = dg_{f(x_0)} \circ df_{x_0}$$

*(composition of linear transformations)*

$$DF(x_0) = Dg(f(x_0))Df(x_0)$$

*(matrix multiplication)*

**Remark:** The statement is exactly the same as in the univariate case, except we replace the univariate derivative by a linear transformation. The proof is more or less the same, with a bit of linear algebra added.

# Mean Value Theorem

**Theorem 4** (Thm. 1.7, Mean Value Theorem, Univariate Case).

$a < b$  Let  $a, b \in \mathbb{R}$ . Suppose  $f : [a, b] \rightarrow \mathbb{R}$  is continuous on  $[a, b]$  and differentiable on  $(a, b)$ . Then there exists  $c \in (a, b)$  such that

$$\frac{f(b) - f(a)}{b - a} = f'(c)$$

that is, such that

$$f(b) - f(a) = f'(c)(b - a)$$

*Proof.* Consider the function  $g : [a, b] \rightarrow \mathbb{R}$

$$g(x) = f(x) - f(a) - \frac{f(b) - f(a)}{b - a}(x - a)$$

Note  $g(a) = g(b) = 0$ .

Then  $g(a) = 0 = g(b)$ . Note that for  $x \in (a, b)$ ,

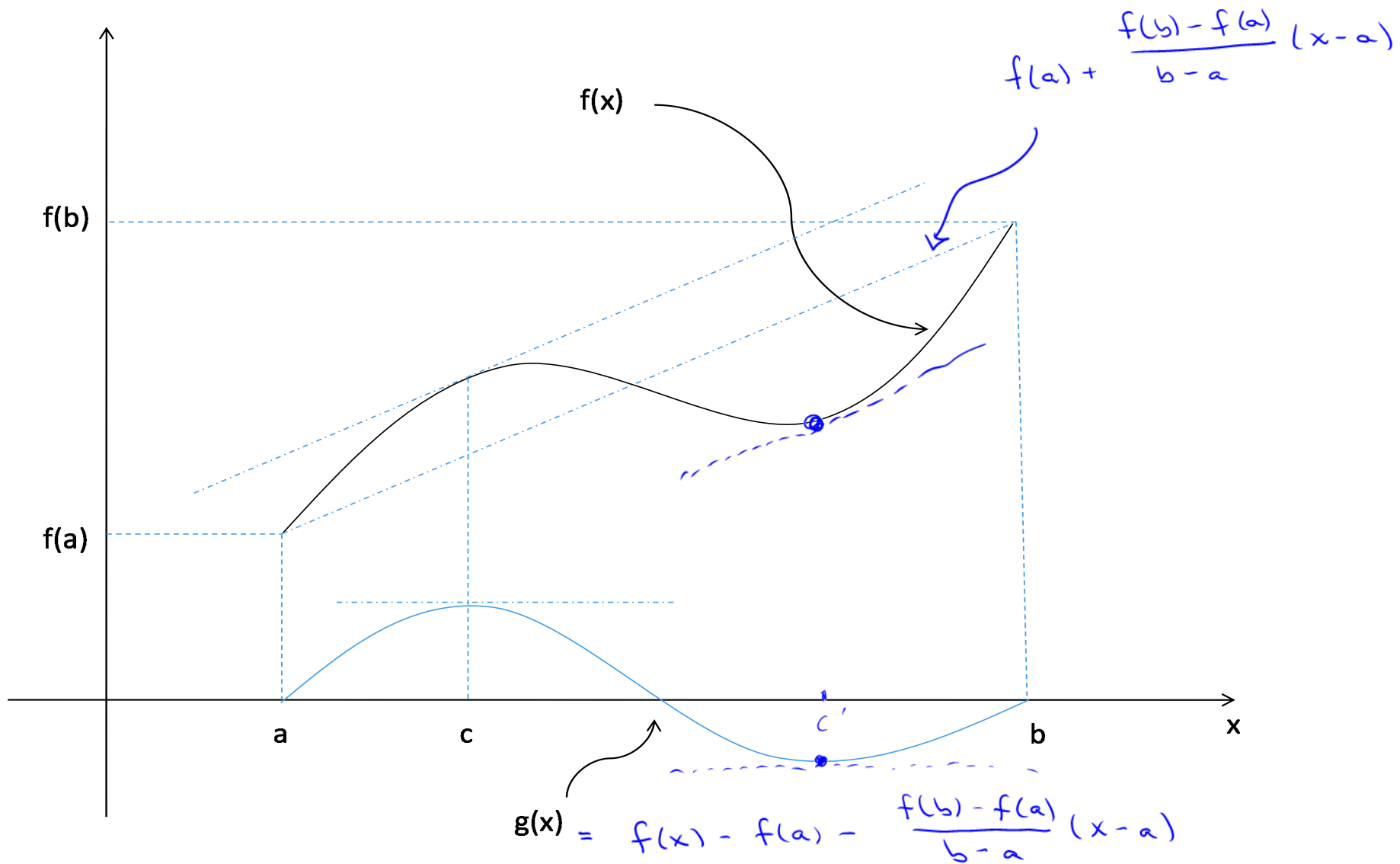
$$g'(x) = f'(x) - \frac{f(b) - f(a)}{b - a}$$

so it suffices to find  $c \in (a, b)$  such that  $g'(c) = 0$ .

Case I: If  $g(x) = 0$  for all  $x \in [a, b]$ , choose an arbitrary  $c \in (a, b)$ , and note that  $g'(c) = 0$ , so we are done.

Case II: Suppose  $g(x) > 0$  for some  $x \in [a, b]$ . Since  $g$  is continuous on  $[a, b]$ , it attains its maximum at some point  $c \in (a, b)$ . Since  $g$  is differentiable at  $c$  and  $c$  is an interior point of the domain of  $g$ , we have  $g'(c) = 0$ , and we are done.

Case III: If  $g(x) < 0$  for some  $x \in [a, b]$ , the argument is similar to that in Case II. □



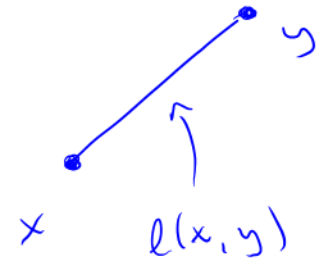


# Mean Value Theorem

**Notation:**

$$\ell(x, y) = \{\alpha x + (1 - \alpha)y : \alpha \in [0, 1]\}$$

is the line segment from  $x$  to  $y$ .  $x, y \in \mathbb{R}^n$



**Theorem 5** (Mean Value Theorem). Suppose  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  is differentiable on an open set  $X \subseteq \mathbb{R}^n$ ,  $x, y \in X$  and  $\ell(x, y) \subseteq X$ . Then there exists  $z \in \ell(x, y)$  such that

$$\begin{aligned} f(y) - f(x) &= Df(z)(y - x) \\ &= df_z(y - x) \end{aligned}$$

Consider  $\bar{f} : [0, 1] \rightarrow \mathbb{R}$  given by  
 $\bar{f}(\alpha) = f(\alpha x + (1 - \alpha)y)$

$$f: \mathbb{R}^n \rightarrow \mathbb{R}^m$$

$$f = (f^1, \dots, f^m), \quad f^i: \mathbb{R}^n \rightarrow \mathbb{R}$$

Notice that the statement is exactly the same as in the univariate case. For  $f: \mathbf{R}^n \rightarrow \mathbf{R}^m$ , we can apply the Mean Value Theorem to each component, to obtain  $z_1, \dots, z_m \in \ell(x, y)$  such that

$$f^i(y) - f^i(x) = Df^i(z_i)(y - x)$$

However, we cannot <sup>always</sup> find a single  $z$  which works for every component.

Note that each  $z_i \in \ell(x, y) \subset \mathbf{R}^n$ ; there are  $m$  of them, one for each component in the range.

# Mean Value Theorem

**Theorem 6.** Suppose  $X \subset \mathbf{R}^n$  is open and  $f : X \rightarrow \mathbf{R}^m$  is differentiable. If  $x, y \in X$  and  $\ell(x, y) \subseteq X$ , then there exists  $z \in \ell(x, y)$  such that

$$\begin{aligned} \|f(y) - f(x)\| &\leq \|df_z(y - x)\| = \|Df(z)(y - x)\| \\ &\leq \|df_z\| \|y - x\| \end{aligned}$$

norm on linear transformations from Friday  $\leq \|Df(z)\| \|y - x\|$

# Mean Value Theorem

**Remark:** To understand why we don't get equality, consider  $f : [0, 1] \rightarrow \mathbf{R}^2$  defined by

$$f(t) = (\cos 2\pi t, \sin 2\pi t)$$

$f$  maps  $[0, 1]$  to the unit circle in  $\mathbf{R}^2$ . Note that  $f(0) = f(1) = (1, 0)$ , so  $|f(1) - f(0)| = 0$ . However, for any  $z \in [0, 1]$ ,

$$\begin{aligned} |df_z(1 - 0)| &= |2\pi(-\sin 2\pi z, \cos 2\pi z)| \\ &= 2\pi\sqrt{\sin^2 2\pi z + \cos^2 2\pi z} \\ &= 2\pi \neq |f(1) - f(0)| = 0 \end{aligned}$$

$$\text{so } f(1) - f(0) \neq df(z) \cdot (1 - 0) \quad \forall z \in [0, 1]$$

## Taylor's Theorem – $\mathbf{R}$

**Theorem 7** (Thm. 1.9, Taylor's Theorem in  $\mathbf{R}$ ). Let  $f : I \rightarrow \mathbf{R}$  be  $n$ -times differentiable, where  $I \subseteq \mathbf{R}$  is an open interval. If  $x, x + h \in I$ , then

$$f(x + h) = f(x) + \sum_{k=1}^{n-1} \frac{f^{(k)}(x)h^k}{k!} + E_n$$

where  $f^{(k)}$  is the  $k^{\text{th}}$  derivative of  $f$  and

$$E_n = \frac{f^{(n)}(x + \lambda h)h^n}{n!} \text{ for some } \lambda \in (0, 1)$$



$n^{\text{th}}$  order error term or  
"remainder"

**Motivation:** Let

$$\begin{aligned} T_n(h) &= f(x) + \sum_{k=1}^n \frac{f^{(k)}(x)h^k}{k!} && \text{\textit{n}^{th} order polynomial in h} \\ &= f(x) + f'(x)h + \frac{f''(x)h^2}{2} + \dots + \frac{f^{(n)}(x)h^n}{n!} \\ T_n(0) &= f(x) \\ T'_n(h) &= f'(x) + f''(x)h + \dots + \frac{f^{(n)}(x)h^{n-1}}{(n-1)!} \\ T'_n(0) &= f'(x) \\ T''_n(h) &= f''(x) + \dots + \frac{f^{(n)}(x)h^{n-2}}{(n-2)!} \\ T''_n(0) &= f''(x) \\ &\vdots \\ T_n^{(n)}(0) &= f^{(n)}(x) \end{aligned}$$

so  $T_n(h)$  is the unique  $n^{\text{th}}$  degree polynomial such that

$$\begin{aligned}T_n(0) &= f(x) \\T'_n(0) &= f'(x) \\&\vdots \\T_n^{(n)}(0) &= f^{(n)}(x)\end{aligned}$$

## Taylor's Theorem – $\mathbf{R}$

**Theorem 8** (Alternate Taylor's Theorem in  $\mathbf{R}$ ). *Let  $f : I \rightarrow \mathbf{R}$  be  $n$  times differentiable, where  $I \subseteq \mathbf{R}$  is an open interval and  $x \in I$ . Then*

$$f(x + h) = f(x) + \sum_{k=1}^n \frac{f^{(k)}(x)h^k}{k!} + o(h^n) \text{ as } h \rightarrow 0$$

*If  $f$  is  $(n + 1)$  times continuously differentiable, then*

$$f(x + h) = f(x) + \sum_{k=1}^n \frac{f^{(k)}(x)h^k}{k!} + O(h^{n+1}) \text{ as } h \rightarrow 0$$

**Remark:** The first equation in the statement of the theorem is essentially a restatement of the definition of the  $n^{\text{th}}$  derivative. The second statement is proven from Theorem 1.9, and the continuity of the derivative.



## $C^k$ Functions

**Definition 3.** Let  $X \subseteq \mathbf{R}^n$  be open. A function  $f : X \rightarrow \mathbf{R}^m$  is continuously differentiable on  $X$  if

- $f$  is differentiable on  $X$  and
- $df_x$  is a continuous function of  $x$  from  $X$  to  $L(\mathbf{R}^n, \mathbf{R}^m)$ , with respect to the operator norm  $\|df_x\|$

$f$  is  $C^k$  if all partial derivatives of order  $\leq k$  exist and are continuous in  $X$ .

## $C^k$ Functions

**Theorem 9** (Thm. 4.3). *Suppose  $X \subseteq \mathbf{R}^n$  is open and  $f : X \rightarrow \mathbf{R}^m$ . Then  $f$  is continuously differentiable on  $X$  if and only if  $f$  is  $C^1$ .*

## Taylor's Theorem – Linear Terms

**Theorem 10.** *Suppose  $X \subseteq \mathbf{R}^n$  is open and  $x \in X$ . If  $f : X \rightarrow \mathbf{R}^m$  is differentiable, then*

$$f(x + h) = f(x) + Df(x)h + o(h) \text{ as } h \rightarrow 0$$

This is essentially a restatement of the definition of differentiability.

## Taylor's Theorem – Linear Terms

**Theorem 11** (Corollary of 4.4). *Suppose  $X \subseteq \mathbf{R}^n$  is open and  $x \in X$ . If  $f : X \rightarrow \mathbf{R}^m$  is  $C^2$ , then*

$$f(x + h) = f(x) + Df(x)h + O(|h|^2) \text{ as } h \rightarrow 0$$

# Taylor's Theorem – Quadratic Terms

We treat each component of the function separately, so consider  $f : X \rightarrow \mathbf{R}$ ,  $X \subseteq \mathbf{R}^n$  an open set. Let

$$D^2 f(x) = \begin{pmatrix} \frac{\partial^2 f}{\partial x_1^2}(x) & \frac{\partial^2 f}{\partial x_2 \partial x_1}(x) & \cdots & \frac{\partial^2 f}{\partial x_n \partial x_1}(x) \\ \frac{\partial^2 f}{\partial x_1 \partial x_2}(x) & \frac{\partial^2 f}{\partial x_2^2}(x) & \cdots & \frac{\partial^2 f}{\partial x_n \partial x_2}(x) \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial^2 f}{\partial x_1 \partial x_n}(x) & \cdots & \cdots & \frac{\partial^2 f}{\partial x_n^2}(x) \end{pmatrix}$$

$$f \in C^2 \Rightarrow \frac{\partial^2 f}{\partial x_i \partial x_j}(x) = \frac{\partial^2 f}{\partial x_j \partial x_i}(x)$$

$\Rightarrow D^2 f(x)$  is symmetric

$\Rightarrow D^2 f(x)$  has eigenvectors that are an orthonormal basis and thus can be diagonalized

## Taylor's Theorem – Quadratic Terms

**Theorem 12** (Stronger Version of Thm. 4.4). *Let  $X \subseteq \mathbf{R}^n$  be open,  $f : X \rightarrow \mathbf{R}$ ,  $f \in C^2(X)$ , and  $x \in X$ . Then*

$$f(x + h) = f(x) + Df(x)h + \frac{1}{2}h^\top (D^2f(x))h + o(|h|^2) \text{ as } h \rightarrow 0$$

*If  $f \in C^3$ ,*

$$f(x + h) = f(x) + Df(x)h + \frac{1}{2}h^\top (D^2f(x))h + O(|h|^3) \text{ as } h \rightarrow 0$$

$$f: \mathbb{R}^n \rightarrow \mathbb{R}$$

## Characterizing Critical Points

**Definition 4.** We say  $f$  has a saddle at  $x$  if  $Df(x) = 0$  but  $f$  has neither a local maximum nor a local minimum at  $x$ .

$f$  has a critical point at  $x$  if  $Df(x) = 0$ .

## Characterizing Critical Points

**Corollary 1.** Suppose  $X \subseteq \mathbf{R}^n$  is open and  $x \in X$ . If  $f : X \rightarrow \mathbf{R}$  is  $C^2$ , there is an orthonormal basis  $\{v_1, \dots, v_n\}$  and corresponding eigenvalues  $\lambda_1, \dots, \lambda_n \in \mathbf{R}$  of  $D^2f(x)$  such that

$$\begin{aligned} f(x+h) &= f(x + \gamma_1 v_1 + \dots + \gamma_n v_n) && h = \sum_i \gamma_i v_i \\ &= f(x) + \sum_{i=1}^n (Df(x)v_i) \gamma_i + \frac{1}{2} \sum_{i=1}^n \lambda_i \gamma_i^2 + o(|\gamma|^2) \end{aligned}$$

where  $\gamma_i = h \cdot v_i$ .  $= f(x) + Df(x)h + \frac{1}{2} h^T D^2f(x)h + o(|h|^2)$

1. If  $f \in C^3$ , we may strengthen  $o(|\gamma|^2)$  to  $O(|\gamma|^3)$ .

2. If  $f$  has a local maximum or local minimum at  $x$ , then

$$Df(x) = 0$$



3. If  $Df(x) = 0$ , then

- $\lambda_1, \dots, \lambda_n > 0 \Rightarrow f$  has a local minimum at  $x$
- $\lambda_1, \dots, \lambda_n < 0 \Rightarrow f$  has a local maximum at  $x$
- $\lambda_i < 0$  for some  $i$ ,  $\lambda_j > 0$  for some  $j \Rightarrow f$  has a saddle at  $x$
- $\lambda_1, \dots, \lambda_n \geq 0$ ,  $\lambda_i > 0$  for some  $i \Rightarrow f$  has a local minimum or a saddle at  $x$
- $\lambda_1, \dots, \lambda_n \leq 0$ ,  $\lambda_i < 0$  for some  $i \Rightarrow f$  has a local maximum or a saddle at  $x$
- $\lambda_1 = \dots = \lambda_n = 0$  gives no information.

*Proof.* (Sketch) From our study of quadratic forms, we know the behavior of the quadratic terms is determined by the signs of the eigenvalues. If  $\lambda_i = 0$  for some  $i$ , then we know that the quadratic form arising from the second partial derivatives is identically zero in the direction  $v_i$ , and the higher derivatives will determine the behavior of the function  $f$  in the direction  $v_i$ . For example, if  $f(x) = x^3$ , then  $f'(0) = 0$ ,  $f''(0) = 0$ , but we know that  $f$  has a saddle at  $x = 0$ ; however, if  $f(x) = x^4$ , then again  $f'(0) = 0$  and  $f''(0) = 0$  but  $f$  has a local (and global) minimum at  $x = 0$ . □