Econ 204 2019

Lecture 3

Outline

1. Metric Spaces and Normed Spaces
2. Convergence of Sequences in Metric Spaces
3. Sequences in $\mathbf{R}$ and $\mathbf{R}^n$

Metric Spaces and Metrics

Generalize distance and length notions in $\mathbf{R}^n$

Definition 1. *A* metric space *is a pair* (X, d)*, where* X *is a set and* $d : X \times X \to \mathbf{R}_+$ *a function satisfying* $\mathbb{R}_+ = \{ r \in \mathbb{R} : r \geq 0 \}$

1. $d(x, y) \geq 0,\ d(x, y) = 0 \Leftrightarrow x = y\ \forall x, y \in X$

2. $d(x, y) = d(y, x)\ \forall x, y \in X$

3. triangle inequality*:*

$$d(x, z) \leq d(x, y) + d(y, z) \quad \forall x, y, z \in X$$

$$\begin{tikzcd}
 & y \arrow[dr, "d(y, z)"] & \\
x \arrow[ur, "d(x, y)"] \arrow[rr, "d(x, z)"'] & & z
\end{tikzcd}$$

A function $d : X \times X \to \mathbf{R}_+$ *satisfying 1-3 above is called a* metric *on* X*.*

A metric gives a notion of distance between elements of X.

Normed Spaces and Norms

Definition 2. *Let V be a vector space over $\mathbf{R}$. A* norm *on V is a function $\|\cdot\| : V \to \mathbf{R}_+$ satisfying*

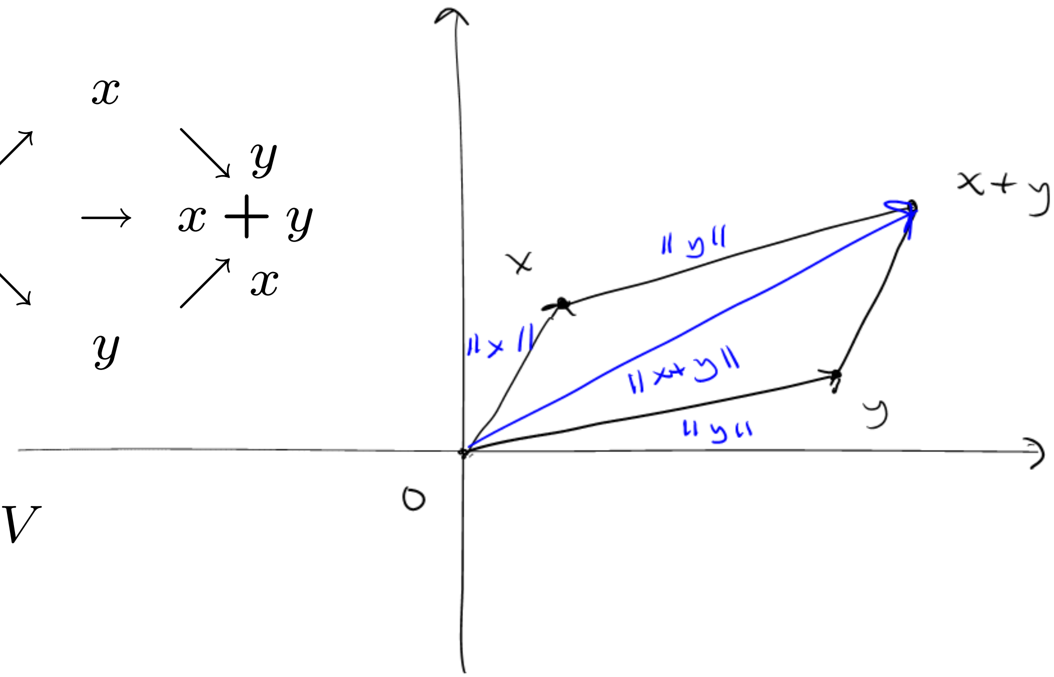
1. $\|x\| \geq 0 \ \forall x \in V$

2. $\|x\| = 0 \Leftrightarrow x = 0 \ \forall x \in V$

3. triangle inequality*:*

$$\|x + y\| \leq \|x\| + \|y\| \ \forall x, y \in V$$

$$\begin{tikzcd} & x \arrow[dr, "y"] & \\ 0 \arrow[ur, "x"] \arrow[rr] \arrow[dr, "y"'] & & x+y \\ & y \arrow[ur, "x"'] & \end{tikzcd}$$



4. $\|\alpha x\| = |\alpha|\|x\| \; \forall \alpha \in \mathbf{R}, x \in V$

A normed vector space *is a vector space over* $\mathbf{R}$ *equipped with a norm.*

A norm gives a notion of length of a vector in V.

$x \in \mathbb{R}^n \qquad x = (x_1, \ldots, x_n) \qquad \|x\| = \left[\sum_{i=1}^{n} x_i^2 \right]^{1/2}$

Normed Spaces and Norms

Example: In $\mathbf{R}^n$, standard notion of distance between two vectors x and y measures length of difference $x - y$, i.e., $d(x, y) = \|x - y\| = \sqrt{\sum_{i=1}^{n} (x_i - y_i)^2}$.

In an abstract normed vector space, the norm can be used analogously to define a notion of distance.

Theorem 1. *Let* $(V, \|\cdot\|)$ *be a normed vector space. Let* $d : V \times V \Rightarrow \mathbf{R}_+$ *be defined by*

$$d(v, w) = \|v - w\|$$

Then (V, d) *is a metric space.*

$(x, y) = (r, s) \Rightarrow x + y = r + s$

Proof. We must verify that d satisfies all the properties of a metric.

1. Let $v, w \in V$. Then by definition, $d(v, w) = \|v - w\| \geq 0$ (why?), and

$$\begin{aligned} d(v,w) = 0 &\Leftrightarrow \|v - w\| = 0 \\ &\Leftrightarrow v - w = 0 \\ &\Leftrightarrow (v + (-w)) + w = w \\ &\Leftrightarrow v + ((-w) + w) = w \\ &\Leftrightarrow v + 0 = w \\ &\Leftrightarrow v = w \end{aligned}$$

vector additive identity

symmetry

2. First, note that for any $x \in V$, $0 \cdot x = (0 + 0) \cdot x = 0 \cdot x + 0 \cdot x$, so $0 \cdot x = 0$. Then $0 = 0 \cdot x = (1 - 1) \cdot x = 1 \cdot x + (-1) \cdot x =$

vector additive identity in V

$= x$

$0 =$ $x + (-1) \cdot x$, so we have $(-1) \cdot x = (-x)$. Then let $v, w \in V$.

$$\begin{aligned} d(v,w) &= \|v - w\| \\ &= |-1|\|v - w\| \\ &= \|(-1)(v + (-w))\| \\ &= \|(-1)v + (-1)(-w)\| \\ &= \|-v + w\| \\ &= \|w + (-v)\| \\ &= \|w - v\| \\ &= d(w,v) \end{aligned}$$

triangle inequality

3. Let $u, w, v \in V$.

$$\begin{aligned} d(u,w) &= \|u-w\| \\ &= \|u+(-v+v)-w\| \\ &= \|(u-v)+(v-w)\| \\ &\leq \|u-v\|+\|v-w\| \\ &= d(u,v)+d(v,w) \end{aligned}$$

Thus d is a metric on V. $\square$

Normed Spaces and Norms

Examples

$\mathbb{R}^n$ with standard norm

- $\mathbf{E}^n$: n-dimensional Euclidean space.

$$V = \mathbf{R}^n, \ \|x\|_2 = |x| = \sqrt{\sum_{i=1}^{n} (x_i)^2}$$

- $V = \mathbf{R}^n$, $\|x\|_1 = \sum_{i=1}^{n} |x_i|$ (the "taxi cab" norm or L^1 norm)

- $V = \mathbf{R}^n$, $\|x\|_\infty = \max\{|x_1|, \ldots, |x_n|\}$ (the maximum norm, or sup norm, or L^∞ norm)

Recall: $C([0,1])$: continuous functions $f:[0,1]\to\mathbb{R}$

$V=$

- $C([0,1]),\ \|f\|_\infty = \sup\{|f(t)| : t\in[0,1]\}$

$V=$

- $C([0,1]),\ \|f\|_2 = \sqrt{\int_0^1 (f(t))^2\,dt}$

$V=$

- $C([0,1]),\ \|f\|_1 = \int_0^1 |f(t)|\,dt$

$v, w \in \mathbb{R}^n \qquad \langle v, w \rangle = v \cdot w = \sum_{i=1}^{n} v_i w_i$

Normed Spaces and Norms

Theorem 2 (Cauchy-Schwarz Inequality).
If $v, w \in \mathbf{R}^n$, *then*

$$\left(\sum_{i=1}^{n} v_i w_i\right)^2 \leq \left(\sum_{i=1}^{n} v_i^2\right)\left(\sum_{i=1}^{n} w_i^2\right)$$

$$|\langle v, w \rangle|^2 = |v \cdot w|^2 \leq |v|^2 |w|^2 = \|v\|^2 \|w\|^2$$

$$|\langle v, w \rangle| = |v \cdot w| \leq |v||w| = \|v\| \|w\|$$

- learn some proof
- triangle inequality of $\|\cdot\|_2$ in $\mathbb{R}^n$ follows from C-S inequality (nice exercise)

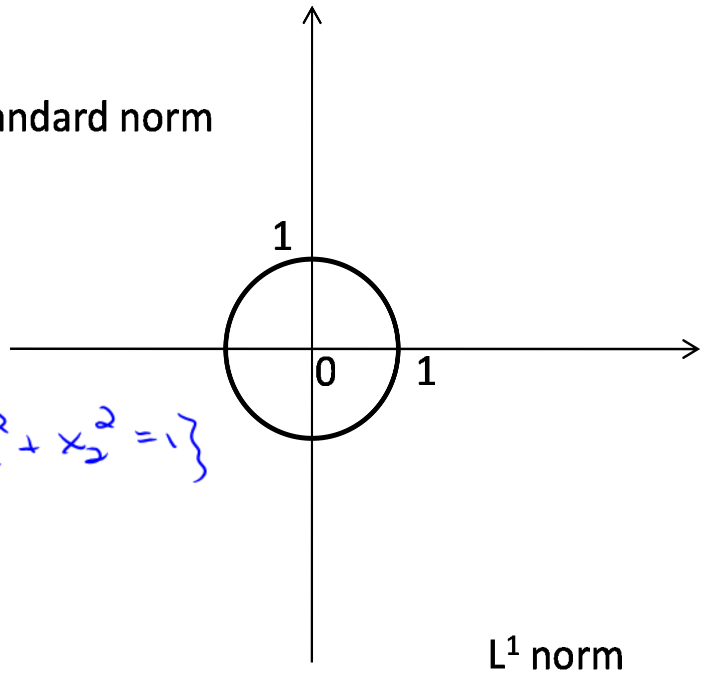
Equivalent Norms

A given vector space may have many different norms: if $\|\cdot\|$ is a norm on a vector space V, so are $2\|\cdot\|$ and $3\|\cdot\|$ and $k\|\cdot\|$ for any $k > 0$.

Less trivially, $\mathbf{R}^n$ supports many different norms as in the examples above. Different norms on a given vector space yield different geometric properties.

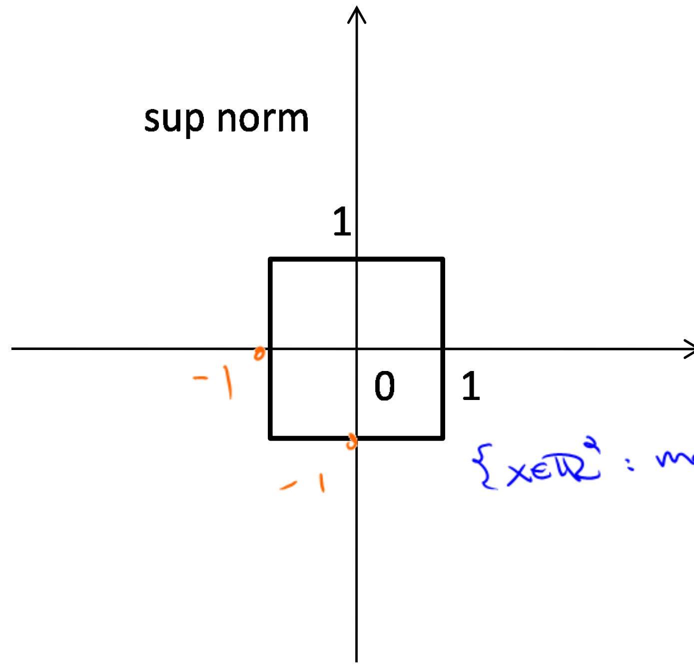
$\{x \in \mathbb{R}^2 : \|x\| = 1\}$ for different norms:

standard norm



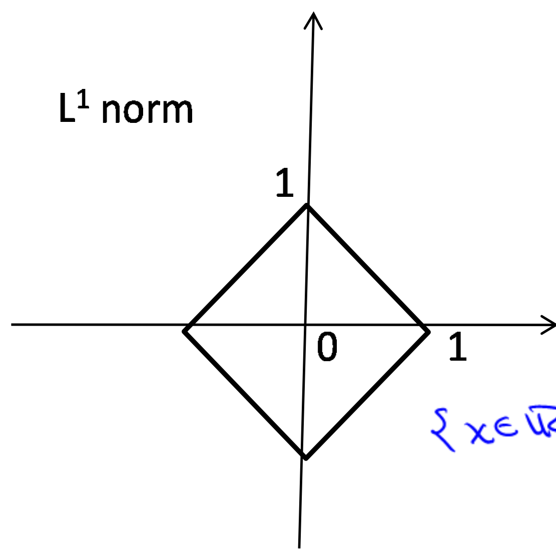
$\{x \in \mathbb{R}^2 : x_1^2 + x_2^2 = 1\}$

sup norm



$\{x \in \mathbb{R}^2 : \max(|x_1|, |x_2|) = 1\}$

L^1 norm



$\{x \in \mathbb{R}^2 : |x_1| + |x_2| = 1\}$

unit balls around 0 in different norms

Equivalent Norms

Definition 3. *Two norms* $\|\cdot\|$ *and* $\|\cdot\|^*$ *on the same vector space* V *are said to be* Lipschitz-equivalent *(or* equivalent *) if* $\exists m, M > 0$ *s.t.* $\forall x \in V$,

$$m\|x\| \le \|x\|^* \le M\|x\|$$

Equivalently, $\exists m, M > 0$ *s.t.* $\forall x \in V, x \neq 0$,

$$m \le \frac{\|x\|^*}{\|x\|} \le M$$

this is an equivalence relation (nice exercise)

Equivalent Norms

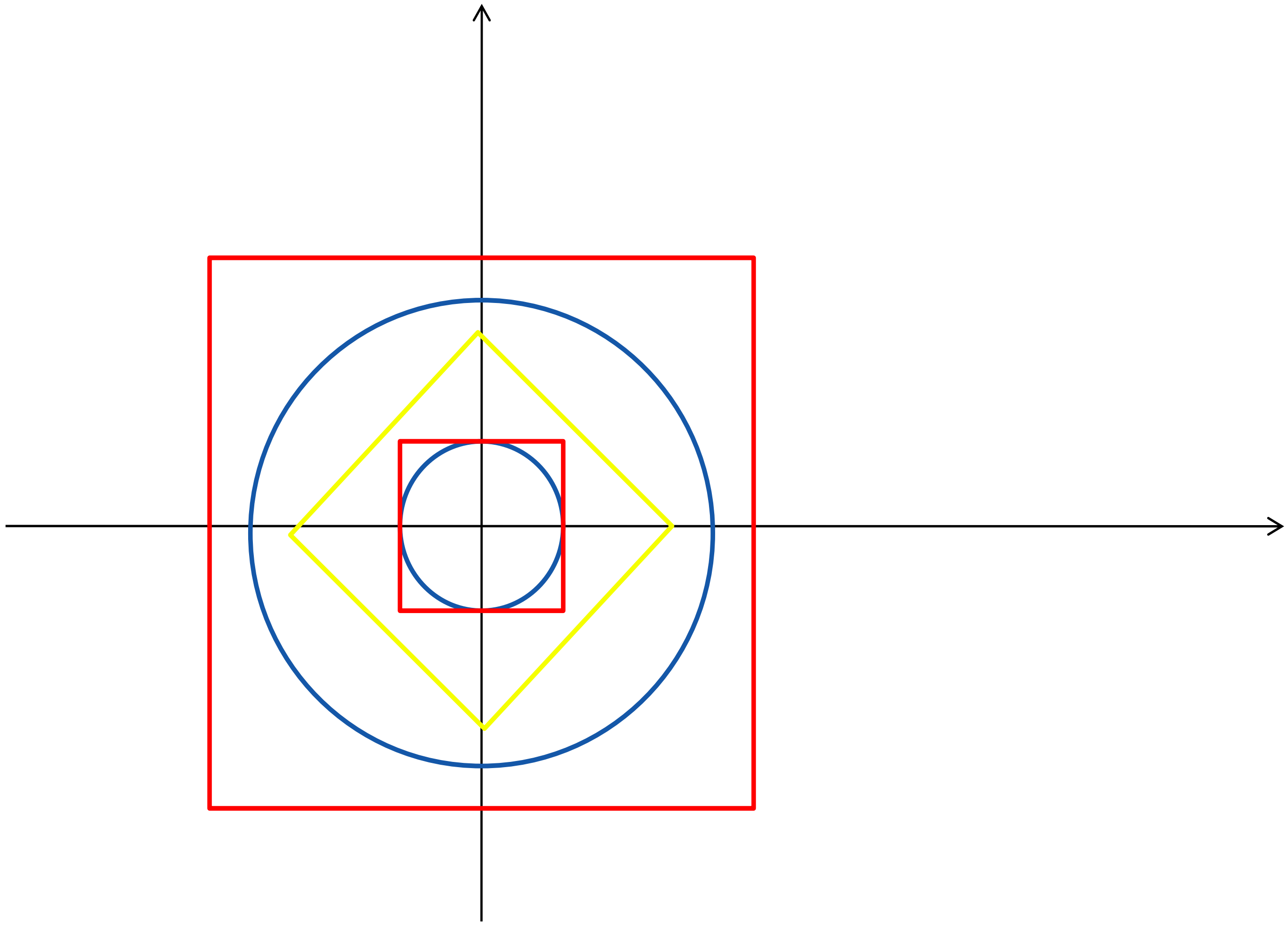
If two norms are equivalent, then they define the same notions of convergence and continuity. For topological purposes, equivalent norms are indistinguishable.

For example, suppose two norms $\|\cdot\|$ and $\|\cdot\|^*$ on the vector space V are equivalent, and fix $x \in V$. Let

$$\begin{aligned} B_\varepsilon(x, \|\cdot\|) &= \{y \in V : \|x-y\| < \varepsilon\} \\ B_\varepsilon(x, \|\cdot\|^*) &= \{y \in V : \|x-y\|^* < \varepsilon\} \end{aligned}$$

Then for any $\varepsilon > 0$,

$$B_{\frac{\varepsilon}{M}}(x, \|\cdot\|) \subseteq B_\varepsilon(x, \|\cdot\|^*) \subseteq B_{\frac{\varepsilon}{m}}(x, \|\cdot\|)$$



norms on $\mathbf{R}^n$ are equivalent

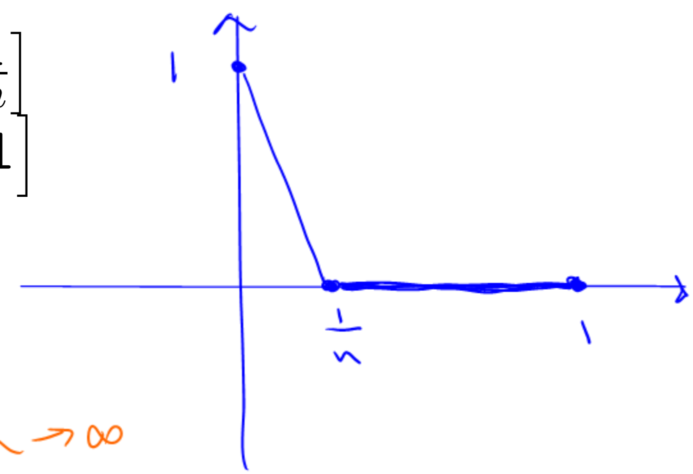
Equivalent Norms

In $\mathbf{R}^n$ (or any finite-dimensional normed vector space), all norms are equivalent. Roughly, up to a difference in scaling, for topological purposes there is a unique norm in $\mathbf{R}^n$.

Theorem 3. *All norms on $\mathbf{R}^n$ are equivalent.*

Infinite-dimensional spaces support norms that are not equivalent. For example, on $C([0,1])$, let f_n be the function

$$f_n(t) = \begin{cases} 1 - nt & \text{if } t \in \left[0, \frac{1}{n}\right] \\ 0 & \text{if } t \in \left(\frac{1}{n}, 1\right] \end{cases}$$



Then

$$\frac{\|f_n\|_1}{\|f_n\|_\infty} = \frac{\frac{1}{2n}}{1} = \frac{1}{2n} \to 0$$

as $n \to \infty$

$\|f_n\|_1 = \int_0^1 |f_n(t)|\,dt = \frac{1}{2n}$

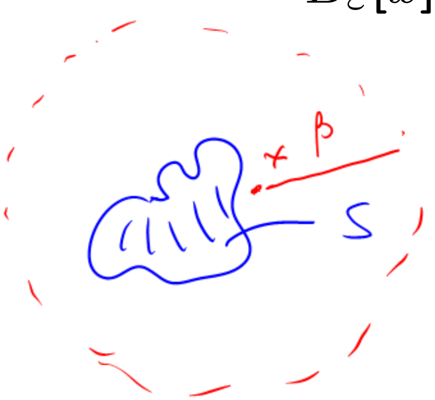
$\|f_n\|_\infty = \sup\{|f_n(t)| : t \in [0,1]\} = 1$

Metrics and Sets

Definition 4. *In a metric space* (X, d)*, a subset* $S \subseteq X$ *is* bounded *if* $\exists x \in X, \beta \in \mathbf{R}$ *such that* $\forall s \in S$, $d(s, x) \leq \beta$.

In a metric space (X, d), define for $\varepsilon > 0$

$$
\begin{aligned}
B_\varepsilon(x) &= \{y \in X : d(y, x) < \varepsilon\} \\
&= \text{"\textit{open ball}" \textit{with center} } x \text{ \textit{and radius} } \varepsilon \\
B_\varepsilon[x] &= \{y \in X : d(y, x) \leq \varepsilon\} \\
&= \text{"\textit{closed ball}" \textit{with center} } x \text{ \textit{and radius} } \varepsilon
\end{aligned}
$$

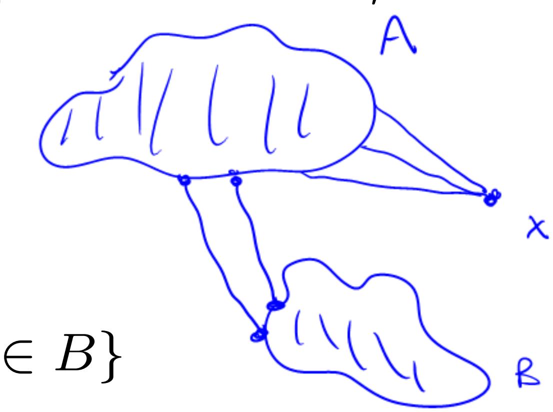


Metrics and Sets

We can use the metric d to define a generalization of "radius". In a metric space (X, d), define the *diameter* of a subset $S \subseteq X$ by

$$\text{diam } (S) = \sup\{d(s, s') : s, s' \in S\}$$

Similarly, we can define the distance from a point to a set, and distance between sets, as follows:



$$\begin{aligned} d(A, x) &= \inf_{a \in A} d(a, x) \\ d(A, B) &= \inf_{a \in A} d(B, a) \\ &= \inf\{d(a, b) : a \in A, b \in B\} \end{aligned}$$

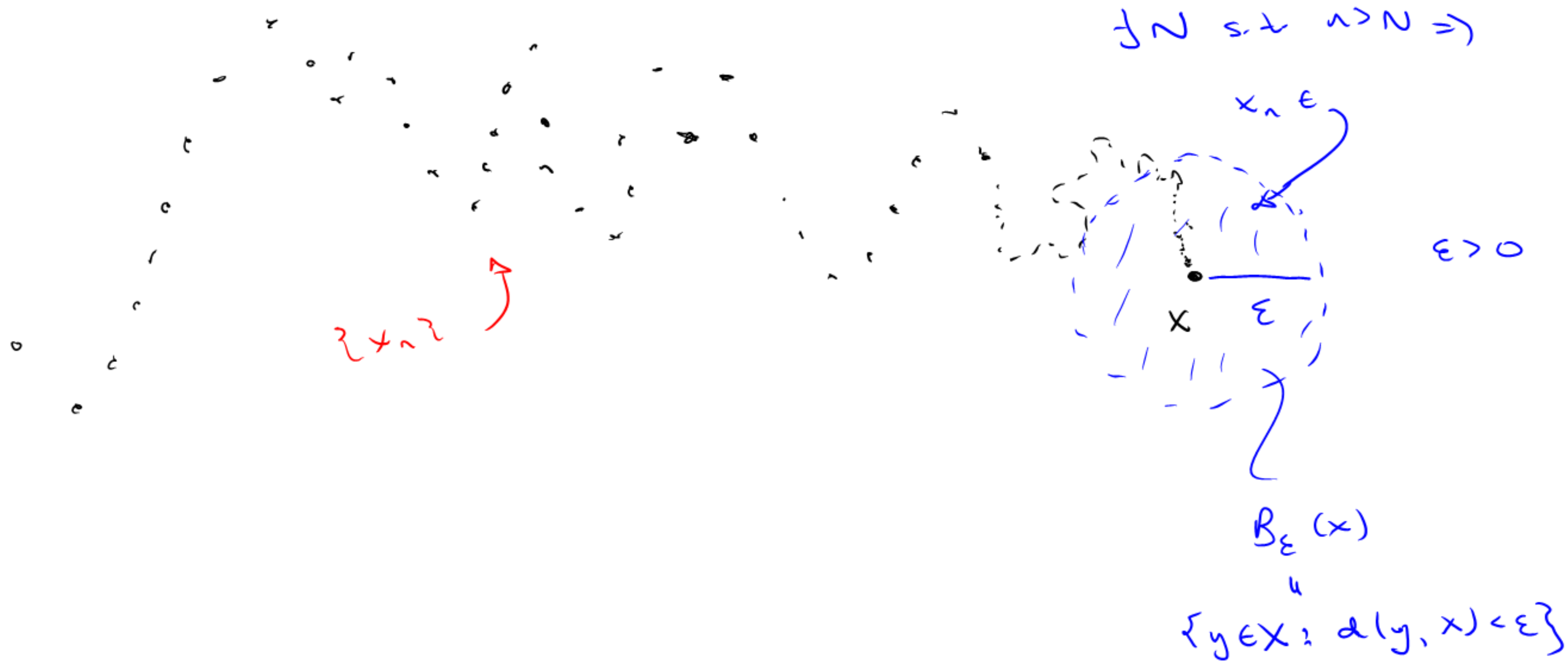
But $d(A, B)$ is **not** a metric.

Convergence of Sequences

Definition 5. *Let* (X, d) *be a metric space. A sequence* $\{x_n\}$ converges *to* x *(written* $x_n \to x$ *or* $\lim_{n\to\infty} x_n = x$*) if*

$$\forall \varepsilon > 0 \ \exists N(\varepsilon) \in \mathbf{N} \text{ s.t. } \forall \ n > N(\varepsilon) \cancel{\Rightarrow} d(x_n, x) < \varepsilon$$

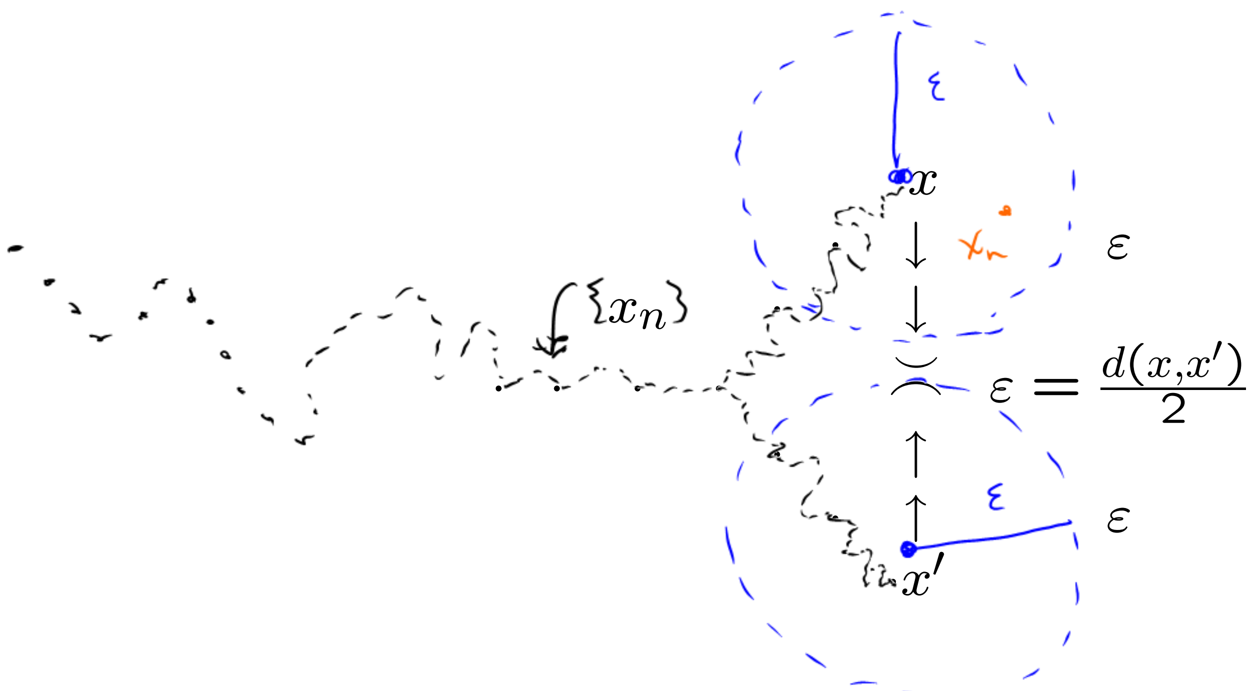
Notice that this is exactly the same as the definition of convergence of a sequence of real numbers, except we replace the standard measure of distance $|\cdot|$ in $\mathbf{R}$ by the general metric d.



$\{x_n\}$
$\exists N$ s.t. $n > N \Rightarrow$
$x_n \in$
$\varepsilon > 0$
x
ε
$B_\varepsilon(x)$
$\shortparallel$
$\{y \in X : d(y, x) < \varepsilon\}$

Uniqueness of Limits

Theorem 4 (Uniqueness of Limits)**.** *In a metric space* (X, d)*, if* $x_n \to x$ *and* $x_n \to x'$*, then* $x = x'$*.*



Proof. Suppose $\{x_n\}$ is a sequence in X, $x_n \to x$, $x_n \to x'$, $x \neq x'$.

Since $x \neq x'$, $d(x, x') > 0$. Let

$$\varepsilon = \frac{d(x, x')}{2} > 0$$

Then there exist $N(\varepsilon)$ and $N'(\varepsilon)$ such that

$$n > N(\varepsilon) \;\Rightarrow\; d(x_n, x) < \varepsilon \qquad (x_n \to x)$$

$$n > N'(\varepsilon) \;\Rightarrow\; d(x_n, x') < \varepsilon \qquad (x_n \to x')$$

Choose

$$n > \max\{N(\varepsilon), N'(\varepsilon)\}$$

Fix $n > \max\{N, N'\}$

Then

$$\begin{aligned} d(x,x') &\leq d(x,x_n) + d(x_n,x') \\ &< \varepsilon + \varepsilon \\ &= 2\varepsilon \\ &= d(x,x') \\ \Rightarrow d(x,x') &< d(x,x') \end{aligned}$$

a contradiction. $\square$

Cluster Points

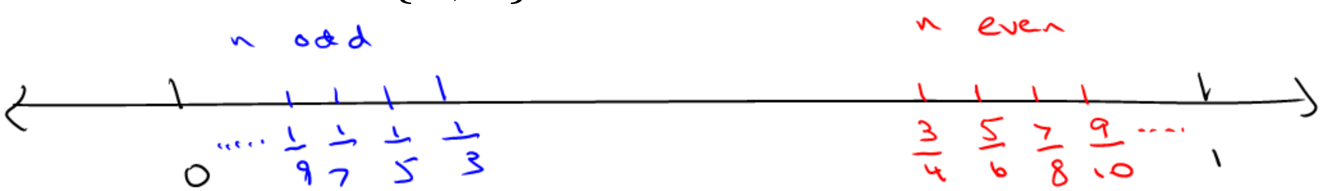
Definition 6. *An element* c *is a* cluster point *of a sequence* $\{x_n\}$ *in a metric space* (X, d) *if* $\forall \varepsilon > 0$, $\{n : x_n \in B_\varepsilon(c)\}$ *is an infinite set. Equivalently,*

$$\forall \varepsilon > 0, N \in \mathbf{N} \ \exists n > N \ \textit{s.t.} \ x_n \in B_\varepsilon(c)$$

Example:

$$x_n = \begin{cases} 1 - \frac{1}{n} & \text{if } n \text{ even} \\ \frac{1}{n} & \text{if } n \text{ odd} \end{cases}$$

For n large and odd, x_n is close to zero; for n large and even, x_n is close to one. The sequence does not converge; the set of cluster points is $\{0, 1\}$.



Subsequences

If $\{x_n\}$ is a sequence and $n_1 < n_2 < n_3 < \cdots$ then $\{x_{n_k}\}$ is called a *subsequence*. of $\{x_n\}$

Note that a subsequence is formed by taking some of the elements of the parent sequence, *in the same order*.

Example: $x_n = \frac{1}{n}$, so $\{x_n\} = \left(1, \frac{1}{2}, \frac{1}{3}, \ldots\right)$. If $n_k = 2k$, then $\{x_{n_k}\} = \left(\frac{1}{2}, \frac{1}{4}, \frac{1}{6}, \ldots\right)$.

$n=1$, $n=2$, $n=3$, - - - -

Cluster Points and Subsequences

Theorem 5 (2.4 in De La Fuente, plus ...). *Let* (X, d) *be a metric space,* $c \in X$*, and* $\{x_n\}$ *a sequence in* X*. Then* c *is a cluster point of* $\{x_n\}$ *if and only if there is a subsequence* $\{x_{n_k}\}$ of $\{x_n\}$ *such that* $\lim_{k\to\infty} x_{n_k} = c$*.*

$\Rightarrow$: *Proof.* Suppose c is a cluster point of $\{x_n\}$. We inductively construct a subsequence that converges to c. For $k = 1$, $\{n : x_n \in B_1(c)\}$ is infinite, so nonempty; let

$$n_1 = \min\{n : x_n \in B_1(c)\}$$

Now, suppose we have chosen $n_1 < n_2 < \cdots < n_k$ such that

$$x_{n_j} \in B_{\frac{1}{j}}(c) \text{ for } j = 1, \ldots, k$$

$\{n : x_n \in B_{\frac{1}{k+1}}(c)\}$ is infinite, so it contains at least one element bigger than n_k, so let

$$n_{k+1} = \min\left\{n : n > n_k,\ x_n \in B_{\frac{1}{k+1}}(c)\right\}$$

Thus, we have chosen $n_1 < n_2 < \cdots < n_k < n_{k+1}$ such that

$$x_{n_j} \in B_{\frac{1}{j}}(c) \text{ for } j = 1, \ldots, k, k+1$$

Thus, by induction, we obtain a subsequence $\{x_{n_k}\}$ such that

$$x_{n_k} \in B_{\frac{1}{k}}(c) \qquad \forall k$$

Given any $\varepsilon > 0$, by the Archimedean property, there exists $N(\varepsilon) > 1/\varepsilon$.

$$\begin{aligned} k > N(\varepsilon) \ &\Rightarrow\ x_{n_k} \in B_{\frac{1}{k}}(c) \qquad k > N \Rightarrow \frac{1}{k} < \varepsilon \\ &\Rightarrow\ x_{n_k} \in B_{\varepsilon}(c) \end{aligned}$$

so

$$x_{n_k} \to c \text{ as } k \to \infty$$

$\Leftarrow$: Conversely, suppose that there is a subsequence $\{x_{n_k}\}$ converging to c. Given any $\varepsilon > 0$, there exists $K \in \mathbf{N}$ such that

$$k > K \Rightarrow d(x_{n_k}, c) < \varepsilon \Rightarrow x_{n_k} \in B_\varepsilon(c)$$

Therefore,

$$\{n : x_n \in B_\varepsilon(c)\} \supseteq \{n_{K+1}, n_{K+2}, n_{K+3}, \ldots\}$$

Since $n_{K+1} < n_{K+2} < n_{K+3} < \cdots$, this set is infinite, so c is a cluster point of $\{x_n\}$. $\square$

Sequences in $\mathbf{R}$ and $\mathbf{R}^m$

Definition 7. *A sequence of real numbers* $\{x_n\}$ *is* increasing *(*decreasing*) if* $x_{n+1} \geq x_n$ *(* $x_{n+1} \leq x_n$ *) for all* n*.*

Definition 8. *If* $\{x_n\}$ *is a sequence of real numbers,* $\{x_n\}$ tends to infinity *(written* $x_n \to \infty$ *or* $\lim x_n = \infty$*) if*

$$\forall K \in \mathbf{R} \ \exists N(K) \text{ s.t. } n > N(K) \Rightarrow x_n > K$$

Similarly define $x_n \to -\infty$ *or* $\lim x_n = -\infty$*.*

Increasing and Decreasing Sequences

Theorem 6 (Theorem 3.1'). *Let* $\{x_n\}$ *be an increasing (decreasing) sequence of real numbers. Then*

$$\lim_{n\to\infty} x_n = \sup\{x_n : n \in \mathbf{N}\}$$

$$\left(\lim_{n\to\infty} x_n = \inf\{x_n : n \in \mathbf{N}\} \right)$$

In particular, the limit exists.

work through proof in dLF - think about unbounded case

Lim Sups and Lim Infs

Consider a sequence $\{x_n\}$ of real numbers. Let

$$\begin{aligned}
\alpha_n &= \sup\{x_k : k \geq n\} \\
&= \sup\{x_n, x_{n+1}, x_{n+2}, \ldots\} \\
\beta_n &= \inf\{x_k : k \geq n\} \\
&= \inf\{x_n, x_{n+1}, x_{n+2}, \ldots\}
\end{aligned}$$

Either $\alpha_n = +\infty$ for all n, or $\alpha_n \in \mathbf{R}$ $\forall n$ and $\alpha_1 \geq \alpha_2 \geq \alpha_3 \geq \cdots$.

Either $\beta_n = -\infty$ for all n, or $\beta_n \in \mathbf{R}$ $\forall n$ and $\beta_1 \leq \beta_2 \leq \beta_3 \leq \cdots$.

Lim Sups and Lim Infs

Definition 9.

$$\limsup_{n\to\infty} x_n = \begin{cases} +\infty & \text{if } \alpha_n = +\infty \text{ for all } n \\ \lim \alpha_n & \text{otherwise.} \end{cases}$$

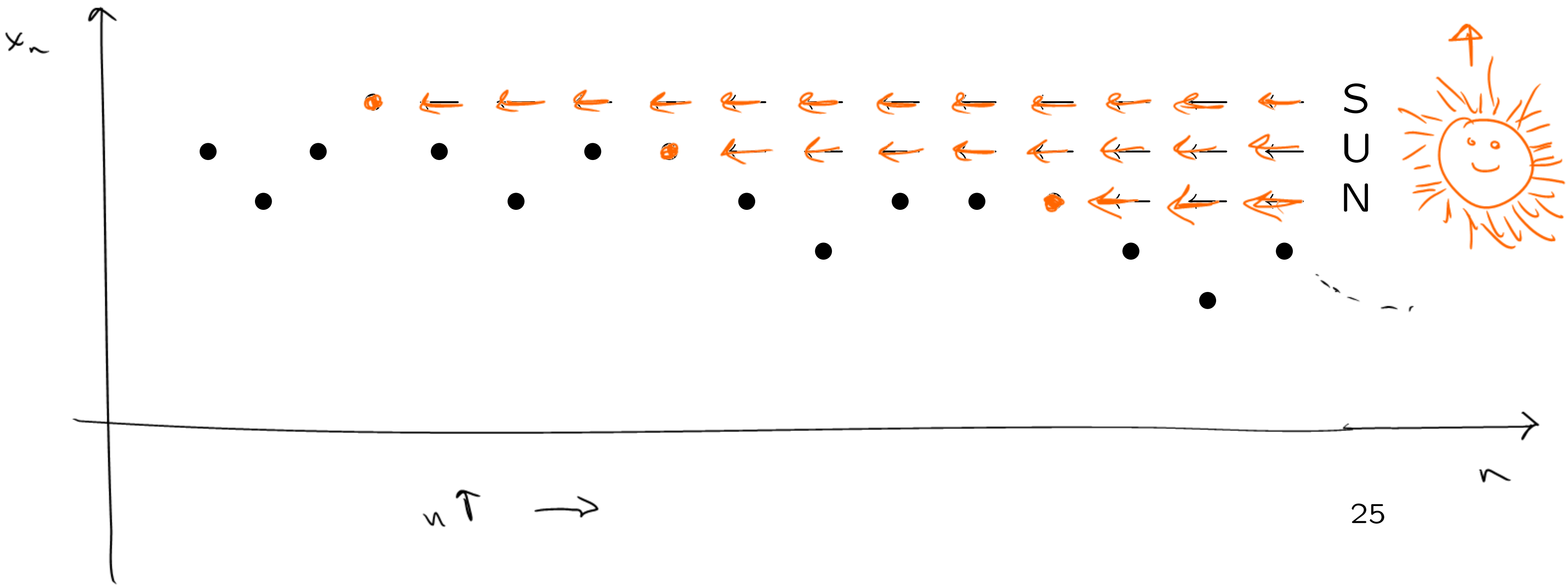
$$\liminf_{n\to\infty} x_n = \begin{cases} -\infty & \text{if } \beta_n = -\infty \text{ for all } n \\ \lim \beta_n & \text{otherwise.} \end{cases}$$

Theorem 7. *Let $\{x_n\}$ be a sequence of real numbers. Then*

$$\lim_{n\to\infty} x_n = \gamma \in \mathbf{R} \cup \{-\infty, \infty\}$$
$$\Leftrightarrow \limsup_{n\to\infty} x_n = \liminf_{n\to\infty} x_n = \gamma$$

Increasing and Decreasing Subsequences

Theorem 8 (Theorem 3.2, Rising Sun Lemma)**.** *Every sequence of real numbers contains an increasing subsequence or a decreasing subsequence or both.*



Proof. Let

$$S = \{s \in \mathbf{N} : x_s > x_n \quad \forall n > s\}$$

Either S is infinite, or S is finite.

If S is infinite, let

$$\begin{aligned} n_1 &= \min S \\ n_2 &= \min (S \setminus \{n_1\}) \\ n_3 &= \min (S \setminus \{n_1, n_2\}) \\ &\vdots \\ n_{k+1} &= \min (S \setminus \{n_1, n_2, \ldots, n_k\}) \end{aligned}$$

Then $n_1 < n_2 < n_3 < \cdots$.

$$\begin{array}{rl} x_{n_1} > x_{n_2} & \text{since } n_1 \in S \text{ and } n_2 > n_1 \\ x_{n_2} > x_{n_3} & \text{since } n_2 \in S \text{ and } n_3 > n_2 \\ & \vdots \\ x_{n_k} > x_{n_{k+1}} & \text{since } n_k \in S \text{ and } n_{k+1} > n_k \\ & \vdots \end{array}$$

so $\{x_{n_k}\}$ is a strictly decreasing subsequence of $\{x_n\}$.

If S is finite and nonempty, let $n_1 = (\max S) + 1$; if $S = \emptyset$, let $n_1 = 1$. Then

$$\begin{array}{rcl} n_1 \notin S & \text{so} & \exists n_2 > n_1 \text{ s.t. } x_{n_2} \geq x_{n_1} \\ n_2 \notin S & \text{so} & \exists n_3 > n_2 \text{ s.t. } x_{n_3} \geq x_{n_2} \\ & \vdots & \\ n_k \notin S & \text{so} & \exists n_{k+1} > n_k \text{ s.t. } x_{n_{k+1}} \geq x_{n_k} \\ & \vdots & \end{array}$$

so $\{x_{n_k}\}$ is a (weakly) increasing subsequence of $\{x_n\}$. $\square$

Bolzano-Weierstrass Theorem

Theorem 9 (Thm. 3.3, Bolzano-Weierstrass)**.** *Every bounded sequence of real numbers contains a convergent subsequence.*

Proof. Let $\{x_n\}$ be a bounded sequence of real numbers. By the Rising Sun Lemma, find an increasing or decreasing subsequence $\{x_{n_k}\}$. If $\{x_{n_k}\}$ is increasing, then by Theorem 3.1',

$$\lim x_{n_k} = \sup\{x_{n_k} : k \in \mathbf{N}\} \leq \sup\{x_n : n \in \mathbf{N}\} < \infty$$

since the sequence is bounded; since the limit is finite, the subsequence converges. Similarly, if the subsequence is decreasing, it converges. $\square$