Econ 204 2019

Lecture 8

Outline

1. Bases
2. Linear Transformations
3. Isomorphisms
Linear Combinations and Spans

**Definition 1.** Let $X$ be a vector space over a field $F$. A linear combination of $x_1, \ldots, x_n \in X$ is a vector of the form

$$y = \sum_{i=1}^{n} \alpha_i x_i$$

where $\alpha_1, \ldots, \alpha_n \in F$.

$\alpha_i$ is the coefficient of $x_i$ in the linear combination.

If $V \subseteq X$, the span of $V$, denoted $\text{span} V$, is the set of all linear combinations of elements of $V$.

A set $V \subseteq X$ spans $X$ if $\text{span} V = X$. 
Linear Dependence and Independence

Definition 2. A set $V \subseteq X$ is linearly dependent if there exist $v_1, \ldots, v_n \in V$ and $\alpha_1, \ldots, \alpha_n \in F$ not all zero such that

$$\sum_{i=1}^{n} \alpha_i v_i = 0$$

A set $V \subseteq X$ is linearly independent if it is not linearly dependent.

Thus $V \subseteq X$ is linearly independent if and only if

$$\sum_{i=1}^{n} \alpha_i v_i = 0, \quad v_i \in V \forall i \Rightarrow \alpha_i = 0 \forall i$$
Bases

Definition 3. A Hamel basis (often just called a basis) of a vector space $X$ is a linearly independent set of vectors in $X$ that spans $X$.

Example: $\{(1, 0), (0, 1)\}$ is a basis for $\mathbb{R}^2$ (this is the standard basis).
Example, cont: $\{(1, 1), (-1, 1)\}$ is another basis for $\mathbb{R}^2$:

Suppose $(x, y) = \alpha(1, 1) + \beta(-1, 1)$ for some $\alpha, \beta \in \mathbb{R}$

- $x = \alpha - \beta$
- $y = \alpha + \beta$
- $x + y = 2\alpha$

$\Rightarrow \alpha = \frac{x + y}{2}$

- $y - x = 2\beta$

$\Rightarrow \beta = \frac{y - x}{2}$

$(x, y) = \frac{x + y}{2}(1, 1) + \frac{y - x}{2}(-1, 1)$

Since $(x, y)$ is an arbitrary element of $\mathbb{R}^2$, $\{(1, 1), (-1, 1)\}$ spans $\mathbb{R}^2$. If $(x, y) = (0, 0)$,

$$\alpha = \frac{0 + 0}{2} = 0, \quad \beta = \frac{0 - 0}{2} = 0$$
so the coefficients are all zero, so \{(1, 1), (-1, 1)\} is linearly independent. Since it is linearly independent and spans \(\mathbb{R}^2\), it is a basis.

**Example:** \{(1, 0, 0), (0, 1, 0)\} is not a basis of \(\mathbb{R}^3\), because it does not span \(\mathbb{R}^3\).

**Example:** \{(1, 0), (0, 1), (1, 1)\} is not a basis for \(\mathbb{R}^2\).

\[
1(1, 0) + 1(0, 1) + (-1)(1, 1) = (0, 0)
\]

so the set is not linearly independent.
Bases

**Theorem 1 (Thm. 1.2').** Let \( V \) be a Hamel basis for \( X \). Then every vector \( x \in X \) has a unique representation as a linear combination of a finite number of elements of \( V \) (with all coefficients nonzero).

*Proof. Let \( x \in X \). Since \( V \) spans \( X \), we can write

\[
x = \sum_{s \in S_1} \alpha_s v_s
\]

where \( S_1 \) is finite, \( \alpha_s \in F, \ \alpha_s \neq 0 \), and \( v_s \in V \) for each \( s \in S_1 \). Now, suppose

\[
x = \sum_{s \in S_1} \alpha_s v_s = \sum_{s \in S_2} \beta_s v_s
\]

*The unique representation of 0 is 0 = \( \sum_{i \in \emptyset} \alpha_i b_i \).
where $S_2$ is finite, $\beta_s \in F$, $\beta_s \neq 0$, and $v_s \in V$ for each $s \in S_2$. Let $S = S_1 \cup S_2$, and define

$$\alpha_s = 0 \text{ for } s \in S_2 \setminus S_1$$

$$\beta_s = 0 \text{ for } s \in S_1 \setminus S_2$$

Then

$$0 = x - x = \sum_{s \in S_1} \alpha_s v_s - \sum_{s \in S_2} \beta_s v_s = \sum_{s \in S} \alpha_s v_s - \sum_{s \in S} \beta_s v_s = \sum_{s \in S} (\alpha_s - \beta_s) v_s$$

Since $V$ is linearly independent, we must have $\alpha_s - \beta_s = 0$, so $\alpha_s = \beta_s$, for all $s \in S$.

$$s \in S_1 \iff \alpha_s \neq 0 \iff \beta_s \neq 0 \iff s \in S_2$$
so \( S_1 = S_2 \) and \( \alpha_s = \beta_s \) for \( s \in S_1 = S_2 \), so the representation is unique. \( \square \)
Bases

**Theorem 2.** *Every vector space has a Hamel basis.*

*Proof.* The proof uses the Axiom of Choice. Indeed, the theorem is equivalent to the Axiom of Choice. □
A closely related result, from which you can derive the previous result, shows that any linearly independent set $V$ in a vector space $X$ can be extended to a basis of $X$.

**Theorem 3.** If $X$ is a vector space and $V \subseteq X$ is linearly independent, then there exists a linearly independent set $W \subseteq X$ such that

$$V \subseteq W \subseteq \text{span } W = X$$
Bases

**Theorem 4.** Any two Hamel bases of a vector space $X$ have the same cardinality (are numerically equivalent).

*Proof.* The proof depends on the so-called Exchange Lemma, whose idea we sketch. Suppose that $V = \{v_\lambda : \lambda \in \Lambda\}$ and $W = \{w_\gamma : \gamma \in \Gamma\}$ are Hamel bases of $X$. Remove one vector $v_{\lambda_0}$ from $V$, so that it no longer spans (if it did still span, then $v_{\lambda_0}$ would be a linear combination of other elements of $V$, and $V$ would not be linearly independent). If $w_\gamma \in \text{span} (V \setminus \{v_{\lambda_0}\})$ for every $\gamma \in \Gamma$, then since $W$ spans, $V \setminus \{v_{\lambda_0}\}$ would also span, contradiction. Thus, we can choose $\gamma_0 \in \Gamma$ such that

$$w_{\gamma_0} \notin \text{span} \left( V \setminus \{v_{\lambda_0}\} \right)$$
Because \( w_{\gamma_0} \in \text{span } V \), we can write

\[
w_{\gamma_0} = \sum_{i=0}^{n} \alpha_i v_{\lambda_i}
\]

where \( \alpha_0 \), the coefficient of \( v_{\lambda_0} \), is not zero (if it were, then we would have \( w_{\gamma_0} \in \text{span } (V \setminus \{v_{\lambda_0}\}) \)). Since \( \alpha_0 \neq 0 \), we can solve for \( v_{\lambda_0} \) as a linear combination of \( w_{\gamma_0} \) and \( v_{\lambda_1}, \ldots, v_{\lambda_n} \), so

\[
\text{span } \left( (V \setminus \{v_{\lambda_0}\}) \cup \{w_{\gamma_0}\} \right) \\
\supseteq \text{span } V \\
= \ X
\]

so

\[
\left( (V \setminus \{v_{\lambda_0}\}) \cup \{w_{\gamma_0}\} \right)
\]

spans \( X \). From the fact that \( w_{\gamma_0} \not\in \text{span } (V \setminus \{v_{\lambda_0}\}) \) one can
show that

\[
\left( (V \setminus \{v_{\lambda_0}\}) \cup \{w_{\gamma_0}\} \right)
\]

is linearly independent, so it is a basis of \( X \). Repeat this process to exchange every element of \( V \) with an element of \( W \) (when \( V \) is uncountable, this is done by a process called transfinite induction). At the end, we obtain a bijection from \( V \) to \( W \), so that \( V \) and \( W \) are numerically equivalent. \( \square \)
Dimension

**Definition 4.** *The dimension of a vector space* $X$, denoted *dim* $X$, *is the cardinality of any basis of* $X$.

For $V \subseteq X$, $|V|$ denotes the cardinality of the set $V$. 
Dimension

Example: The set of all $m \times n$ real-valued matrices is a vector space over $\mathbb{R}$. A basis is given by

$$\{E_{ij} : 1 \leq i \leq m, 1 \leq j \leq n\}$$

where

$$(E_{ij})_{kl} = \begin{cases} 1 & \text{if } k = i \text{ and } \ell = j \\ 0 & \text{otherwise.} \end{cases}$$

The dimension of the vector space of $m \times n$ matrices is $mn$. 
Dimension and Dependence

**Theorem 5** (Thm. 1.4). Suppose \( \dim X = n \in \mathbb{N} \). If \( V \subseteq X \) and \( |V| > n \), then \( V \) is linearly dependent.
Dimension and Dependence

**Theorem 6** (Thm. 1.5’). Suppose $\dim X = n \in \mathbb{N}$, $V \subseteq X$, and $|V| = n$.

- If $V$ is linearly independent, then $V$ spans $X$, so $V$ is a Hamel basis.

- If $V$ spans $X$, then $V$ is linearly independent, so $V$ is a Hamel basis.
Linear Transformations

**Definition 5.** Let $X$ and $Y$ be two vector spaces over the field $F$. We say $T : X \to Y$ is a linear transformation if

$$T(\alpha_1 x_1 + \alpha_2 x_2) = \alpha_1 T(x_1) + \alpha_2 T(x_2) \quad \forall x_1, x_2 \in X, \alpha_1, \alpha_2 \in F$$

Let $L(X, Y)$ denote the set of all linear transformations from $X$ to $Y$. 
Linear Transformations

**Theorem 7.** $L(X,Y)$ is a vector space over $F$.

**Proof.** First, define linear combinations in $L(X,Y)$ as follows. For $T_1, T_2 \in L(X,Y)$ and $\alpha, \beta \in F$, define $\alpha T_1 + \beta T_2$ by

$$(\alpha T_1 + \beta T_2)(x) = \alpha T_1(x) + \beta T_2(x)$$

We need to show that $\alpha T_1 + \beta T_2 \in L(X,Y)$.

$$(\alpha T_1 + \beta T_2)(\gamma x_1 + \delta x_2)$$

$= \alpha T_1(\gamma x_1 + \delta x_2) + \beta T_2(\gamma x_1 + \delta x_2)$$

$= \alpha (\gamma T_1(x_1) + \delta T_1(x_2)) + \beta (\gamma T_2(x_1) + \delta T_2(x_2))$$

$= \gamma (\alpha T_1(x_1) + \beta T_2(x_1)) + \delta (\alpha T_1(x_2) + \beta T_2(x_2))$$

$= \gamma (\alpha T_1 + \beta T_2)(x_1) + \delta (\alpha T_1 + \beta T_2)(x_2)$$
so $\alpha T_1 + \beta T_2 \in L(X, Y)$.

The rest of the proof involves straightforward checking of the vector space axioms.
Compositions of Linear Transformations

Given $R \in L(X, Y)$ and $S \in L(Y, Z)$, $S \circ R : X \to Z$. We will show that $S \circ R \in L(X, Z)$, that is, the composition of two linear transformations is linear.

\[(S \circ R)(\alpha x_1 + \beta x_2) = S(R(\alpha x_1 + \beta x_2))
= S(\alpha R(x_1) + \beta R(x_2))
= \alpha S(R(x_1)) + \beta S(R(x_2))
= \alpha (S \circ R)(x_1) + \beta (S \circ R)(x_2)\]

so $S \circ R \in L(X, Z)$. 
Kernel and Rank

Definition 6. Let $T \in L(X, Y)$.

- The image of $T$ is $\text{Im} T = T(X)$

- The kernel of $T$ is $\ker T = \{x \in X : T(x) = 0\}$

- The rank of $T$ is $\text{Rank} T = \dim(\text{Im} T)$
Rank-Nullity Theorem

**Theorem 8** (Thms. 2.9, 2.7, 2.6: The Rank-Nullity Theorem). Let $X$ be a finite-dimensional vector space, $T \in L(X,Y)$. Then $\text{Im} \, T$ and $\ker T$ are vector subspaces of $Y$ and $X$ respectively, and

$$\dim X = \dim \ker T + \text{Rank} \, T$$
Kernel and Rank

**Theorem 9** (Thm. 2.13). \( T \in L(X,Y) \) is one-to-one if and only if \( \ker T = \{0\} \).

*Proof.* Suppose \( T \) is one-to-one. Suppose \( x \in \ker T \). Then \( T(x) = 0 \). But since \( T \) is linear, \( T(0) = T(0 \cdot 0) = 0 \cdot T(0) = 0 \). Since \( T \) is one-to-one, \( x = 0 \), so \( \ker T = \{0\} \).

Conversely, suppose that \( \ker T = \{0\} \). Suppose \( T(x_1) = T(x_2) \). Then

\[
T(x_1 - x_2) = T(x_1) - T(x_2) = 0
\]

which says \( x_1 - x_2 \in \ker T \), so \( x_1 - x_2 = 0 \), so \( x_1 = x_2 \). Thus, \( T \) is one-to-one. \( \square \)
Invertible Linear Transformations

**Definition 7.** $T \in L(X,Y)$ is invertible if there exists a function $S : Y \to X$ such that

\[
S(T(x)) = x \quad \forall x \in X \\
T(S(y)) = y \quad \forall y \in Y
\]

Denote $S$ by $T^{-1}$.

Note that $T$ is invertible if and only if it is one-to-one and onto. This is just the condition for the existence of an inverse function. The linearity of the inverse follows from the linearity of $T$. 
Invertible Linear Transformations

**Theorem 10** (Thm. 2.11). If $T \in L(X, Y)$ is invertible, then $T^{-1} \in L(Y, X)$, i.e. $T^{-1}$ is linear.

**Proof.** Suppose $\alpha, \beta \in F$ and $v, w \in Y$. Since $T$ is invertible, there exist unique $v', w' \in X$ such that

\[
T(v') = v \quad T^{-1}(v) = v' \\
T(w') = w \quad T^{-1}(w) = w'.
\]

Then

\[
T^{-1}(\alpha v + \beta w) = T^{-1}(\alpha T(v') + \beta T(w')) \\
= T^{-1}(T(\alpha v' + \beta w')) \\
= \alpha v' + \beta w' \\
= \alpha T^{-1}(v) + \beta T^{-1}(w).
\]
so $T^{-1} \in L(Y, X)$.  

\qed
Linear Transformations and Bases

**Theorem 11** (Thm. 3.2). Let $X$ and $Y$ be two vector spaces over the same field $F$, and let $V = \{v_\lambda : \lambda \in \Lambda\}$ be a basis for $X$. Then a linear transformation $T \in L(X,Y)$ is completely determined by its values on $V$, that is:

1. Given any set $\{y_\lambda : \lambda \in \Lambda\} \subseteq Y$, $\exists T \in L(X,Y)$ s.t.
   \[
   T(v_\lambda) = y_\lambda \quad \forall \lambda \in \Lambda
   \]

2. If $S, T \in L(X,Y)$ and $S(v_\lambda) = T(v_\lambda)$ for all $\lambda \in \Lambda$, then $S = T$. 
Proof. 1. If \( x \in X \), \( x \) has a unique representation of the form

\[
x = \sum_{i=1}^{n} \alpha_i v_{\lambda_i} \quad \alpha_i \neq 0 \quad i = 1, \ldots, n
\]

(Recall that if \( x = 0 \), then \( n = 0 \).) Define

\[
T(x) = \sum_{i=1}^{n} \alpha_i y_{\lambda_i}
\]

Then \( T(x) \in Y \). The verification that \( T \) is linear is left as an exercise.
2. Suppose $S(v_\lambda) = T(v_\lambda)$ for all $\lambda \in \Lambda$. Given $x \in X$,

\[
S(x) = S \left( \sum_{i=1}^{n} \alpha_i v_{\lambda_i} \right) \\
= \sum_{i=1}^{n} \alpha_i S(v_{\lambda_i}) \\
= \sum_{i=1}^{n} \alpha_i T(v_{\lambda_i}) \\
= T \left( \sum_{i=1}^{n} \alpha_i v_{\lambda_i} \right) \\
= T(x)
\]

so $S = T$. \qed
Isomorphisms

**Definition 8.** Two vector spaces $X$ and $Y$ over a field $F$ are isomorphic if there is an invertible $T \in L(X,Y)$.

$T \in L(X,Y)$ is an isomorphism if it is invertible (one-to-one and onto).

Isomorphic vector spaces are essentially indistinguishable as vector spaces.
Isomorphisms

Theorem 12 (Thm. 3.3). Two vector spaces $X$ and $Y$ over the same field are isomorphic if and only if $\dim X = \dim Y$.

Proof. Suppose $X$ and $Y$ are isomorphic, and let $T \in L(X, Y)$ be an isomorphism. Let

$$U = \{u_\lambda : \lambda \in \Lambda\}$$

be a basis of $X$, and let $v_\lambda = T(u_\lambda)$ for each $\lambda \in \Lambda$. Set

$$V = \{v_\lambda : \lambda \in \Lambda\}$$

Since $T$ is one-to-one, $U$ and $V$ have the same cardinality. If
$y \in Y$, then there exists $x \in X$ such that

\[
y = T(x) = T \left( \sum_{i=1}^{n} \alpha_{\lambda_i} u_{\lambda_i} \right)
\]

\[
= \sum_{i=1}^{n} \alpha_{\lambda_i} T(u_{\lambda_i})
\]

\[
= \sum_{i=1}^{n} \alpha_{\lambda_i} v_{\lambda_i}
\]

which shows that $V$ spans $Y$. To see that $V$ is linearly indepen-
dent, suppose

\[ 0 = \sum_{i=1}^{m} \beta_i v \lambda_i \]

\[ = \sum_{i=1}^{m} \beta_i T(u \lambda_i) \]

\[ = T \left( \sum_{i=1}^{m} \beta_i u \lambda_i \right) \]

Since \( T \) is one-to-one, \( \ker T = \{0\} \), so

\[ \sum_{i=1}^{m} \beta_i u \lambda_i = 0 \]

Since \( U \) is a basis, we have \( \beta_1 = \cdots = \beta_m = 0 \), so \( V \) is linearly independent. Thus, \( V \) is a basis of \( Y \); since \( U \) and \( V \) are numerically equivalent, \( \dim X = \dim Y \).
Now suppose $\dim X = \dim Y$. Let

$$U = \{u_\lambda : \lambda \in \Lambda\} \text{ and } V = \{v_\lambda : \lambda \in \Lambda\}$$

be bases of $X$ and $Y$; note we can use the same index set $\Lambda$ for both because $\dim X = \dim Y$. By Theorem 3.2, there is a unique
\[ T \in L(X, Y) \] such that \( T(u_\lambda) = v_\lambda \) for all \( \lambda \in \Lambda \). If \( T(x) = 0 \), then

\[
0 = T(x) = T \left( \sum_{i=1}^{n} \alpha_i u_\lambda_i \right) = \sum_{i=1}^{n} \alpha_i T(u_\lambda_i) = \sum_{i=1}^{n} \alpha_i v_\lambda_i \implies \alpha_1 = \cdots = \alpha_n = 0 \text{ since } V \text{ is a basis} \]

\[ \implies x = 0 \]

\[ \implies \ker T = \{0\} \]

\[ \implies T \text{ is one-to-one} \]
If \( y \in Y \), write \( y = \sum_{i=1}^{m} \beta_i v_{\lambda_i} \). Let

\[
x = \sum_{i=1}^{m} \beta_i u_{\lambda_i}
\]

Then

\[
T(x) = T \left( \sum_{i=1}^{m} \beta_i u_{\lambda_i} \right)
= \sum_{i=1}^{m} \beta_i T(u_{\lambda_i})
= \sum_{i=1}^{m} \beta_i v_{\lambda_i}
= y
\]

so \( T \) is onto, so \( T \) is an isomorphism and \( X, Y \) are isomorphic. \( \square \)