Economics 204 Summer/Fall 2019
Lecture 13–Wednesday August 14, 2019

Section 5.3. Fixed Point Theorems: Brouwer's and Kakutani's

We have already studied fixed points for the very special case of contraction mappings. Here we study them for general functions as well as for correspondences.

Definition 1 Let X be a nonempty set and $f : X \to X$. A point $x^* \in X$ is a *fixed point* of f if $f(x^*) = x^*$.

Aptly named, x^* is a fixed point of f if it is "fixed" by the map f.

Examples:

1. Let $X = \mathbf{R}$ and $f : \mathbf{R} \to \mathbf{R}$ be given by $f(x) = 2x$. Then $x = 0$ is a fixed point of f (and is the unique fixed point of f).
2. Let $X = \mathbf{R}$ and $f : \mathbf{R} \to \mathbf{R}$ be given by $f(x) = x$. Then every point in $\mathbf{R}$ is a fixed point of f (in particular, fixed points need not be unique).
3. Let $X = \mathbf{R}$ and $f : \mathbf{R} \to \mathbf{R}$ be given by $f(x) = x + 1$. Then f has no fixed points.
4. Let $X = [0, 2]$ and $f : X \to X$ be given by $f(x) = \frac{1}{2}(x + 1)$. Then

$$\begin{aligned} f(x) &= \frac{1}{2}(x+1) = x \\ &\iff x + 1 = 2x \\ &\iff x = 1 \end{aligned}$$

   So $x = 1$ is the unique fixed point of f. Notice that f is a contraction (why?), so we already knew that f must have a unique fixed point on $\mathbf{R}$ from the Contraction Mapping Theorem.
5. Let $X = [0, \frac{1}{4}] \cup [\frac{3}{4}, 1]$ and $f : X \to X$ be given by $f(x) = 1 - x$. Then f has no fixed points.
6. Let $X = [-2, 2]$ and $f : X \to X$ be given by $f(x) = \frac{1}{2}x^2$. Then f has two fixed points, $x = 0$ and $x = 2$. If instead $X' = (0, 2)$, then $f : X' \to X'$ but f has no fixed points on X'.
7. Let $X = \{1, 2, 3\}$ and $f : X \to X$ be given by $f(1) = 2, f(2) = 3, f(3) = 1$ (so f is a permutation of X). Then f has no fixed points.

8. Let $X = [0, 2]$ and $f : X \to X$ be given by

$$f(x) = \begin{cases} x + 1 & \text{if } x \leq 1 \\ x - 1 & \text{if } x > 1 \end{cases}$$

Then f has no fixed points.

As the examples illustrate, fixed points need not exist, and if they do, they need not be unique. What distinguishes between the different behavior in these examples? When can we guarantee the existence of a fixed point? The Contraction Mapping Theorem gave us one set of conditions, but these are extremely strong, too strong to be useful in many situations. Also, in many cases we cannot guarantee that the mapping of interest is single-valued, so we are also interested in more general questions regarding fixed points for correspondences.

We start with a simple one-dimensional result.

Theorem 2 *Let $X = [a, b]$ for $a, b \in \mathbf{R}$ with $a < b$ and let $f : X \to X$ be continuous. Then f has a fixed point.*

Proof: Let $g : [a, b] \to \mathbf{R}$ be given by

$$g(x) = f(x) - x$$

If either $f(a) = a$ or $f(b) = b$, we're done. So assume $f(a) > a$ and $f(b) < b$. Then

$$\begin{aligned} g(a) &= f(a) - a > 0 \\ g(b) &= f(b) - b < 0 \end{aligned}$$

g is continuous, so by the Intermediate Value Theorem, $\exists x^* \in (a, b)$ such that $g(x^*) = 0$, that is, such that $f(x^*) = x^*$. See Figures 1 and 2. ■

This is a special case of a much more general and powerful result.

Theorem 3 (Thm. 3.2. Brouwer's Fixed Point Theorem) *Let $X \subseteq \mathbf{R}^n$ be nonempty, compact, and convex, and let $f : X \to X$ be continuous. Then f has a fixed point.*

We will not give a complete proof of the general version of Brouwer's Fixed Point Theorem. There are a variety of ways to prove this, but each requires more heavy machinery than we can feasibly introduce here. Instead, we will give two different but fairly complete sketches that will give you different intuitions for why this theorem is true.[1]

Proof (sketch) 1: Consider the case when the set X is the unit ball in $\mathbf{R}^n$, i.e. $X = B_1[0] = B = \{x \in \mathbf{R}^n : \|x\| \leq 1\}$. This restriction to the domain B is essentially without loss of

[1] See J. Franklin, *Methods of Mathematical Economics*, for an elementary (but long) proof.

generality. Let $f : B \to B$ be a continuous function. For this sketch of the proof, we need to use the following fact. This fact is intuitive and not difficult to visualize, but is itself fairly difficult to prove. Recall that ∂B denotes the boundary of B, so $\partial B = \{x \in \mathbf{R}^n : \|x\| = 1\}$.

Fact: Let B be the unit ball in $\mathbf{R}^n$. Then there is no continuous function $h : B \to \partial B$ such that $h(x') = x'$ for every $x' \in \partial B$.

This fact says that there is no way to construct a continuous function mapping the unit ball onto its boundary in a way that fixes every point on the boundary (such a mapping is called a *retraction*). You can visualize this in two dimensions by thinking of the ball constructed out of a thin disk made of a stretchable material (or Silly Putty, if you remember that), and thinking of this mapping h corresponding to trying to rearrange and stretch the disk, keeping the outside edge fixed, so that it becomes a ring, all without tearing it. This is impossible, which the above result verifies.

Now to establish Brouwer's theorem, suppose, by way of contradiction, that f has no fixed points in B. Thus for every $x \in B$, $x \neq f(x)$. Since x and its image $f(x)$ are distinct points in B for every x, we can carry out the following construction. For each $x \in B$, construct the line segment originating at $f(x)$ and going through x. Let $g(x)$ denote the intersection of this line segment with $\partial B = \{x \in B : \|x\| = 1\}$. See Figure 4 for an illustration. This construction is well-defined (notice that it would not be if there were fixed points of f, as in Figure 5), and gives a continuous function $g : B \to \partial B$. Furthermore, notice that if $x' \in \partial B$, then $x' = g(x')$. Again, see Figure 4. That is, $g|_{\partial B} = \text{ id }$, with id denoting the identity function. Since there are no such functions by the fact above, we have a contradiction. Therefore there exists $x^* \in B$ such that $f(x^*) = x^*$, that is, f has a fixed point in B.

Proof (sketch) 2: For this sketch, we appeal to an important result due to Scarf that gives an efficient algorithm to find approximate fixed points, that is, showing:

$$\forall \varepsilon > 0 \;\; \exists x_\varepsilon^* \in X \text{ s.t. } \; |f(x_\varepsilon^*) - x_\varepsilon^*| < \varepsilon$$

In turn this can be viewed as a computational version of an important combinatorial result known as Sperner's Lemma. Here is a sketch of the idea:

- Suppose X is $n-1$ dimensional. Let X be the simplex
$$X = \left\{ p \in \mathbf{R}^n_+ : \sum_{\ell=1}^{n} p_\ell = 1 \right\}$$
- Triangulate X, i.e. divide X into a set of simplices such that the intersection of any two simplices is either empty or a whole face of both. See Figure 6.
- Notice that for each vertex x in the triangulation, either $f(x) = x$, in which case we are done, or there must exist ℓ such that $f(x)_\ell < x_\ell$ (why?).

  Then label each vertex in the triangulation by
$$L(x) = \min\{\ell : f(x)_\ell < x_\ell\}$$

- Each simplex in the triangulation has n vertices. A simplex is *completely labeled* if its vertices carry each of the labels $1, \ldots, n$ exactly once; it is *almost completely labeled* if its vertices carry the labels $1, \ldots, n-1$ with exactly one of these labels repeated. See Figure 6.

- A simplex which is almost completely labeled has two "doors", the faces opposite the two vertices with repeated labels. The algorithm pivots from one simplex to another by going in one door and alway leaving by the other door. The new simplex must either be completely labeled, in which case the algorithm stops, or it is almost completely labeled and the algorithm continues. See Figure 7.

- One can show that one can never visit the same simplex twice: if you did, there would be a first simplex visited a second time, but then you had to enter it through a door, and you previously used both doors, so some other simplex must be the first simplex visited a second time, contradiction.

- One can show that one cannot exit through a face of the the large simplex. Since there are only finitely many simplices, and you visit each one at most once, you must stop after a finite number of steps at a completely labeled simplex.

- Completely labeled simplices are approximate fixed points:

  - Fix $\varepsilon > 0$. Since X is compact and f is continuous, f is uniformly continuous, so we can find a triangulation fine enough so that for every simplex σ in the triangulation,

  $$x, y \in \sigma \Rightarrow |x - y| < \frac{\varepsilon}{4n} \text{ and } |f(x) - f(y)| < \frac{\varepsilon}{4n}$$

  - Suppose σ is completely labeled. Let its vertices be $v_1, \ldots, v_n$, and assume without loss of generality

  $$L(v_\ell) = \ell$$

  Then

  $$\begin{aligned}
  L(v_1) = 1 &\quad\Rightarrow\quad f(v_1)_1 < (v_1)_1 \\
  L(v_2) = 2 \neq 1 &\quad\Rightarrow\quad f(v_2)_1 \geq (v_2)_1 \\
  &\quad\Rightarrow\quad \exists y_1 \in \sigma \text{ s.t. } f(y_1)_1 = (y_1)_1 \\
  L(v_2) = 2 &\quad\Rightarrow\quad f(v_2)_2 < (v_2)_2 \\
  L(v_3) = 3 \neq 2 &\quad\Rightarrow\quad f(v_3)_2 \geq (v_3)_2 \\
  &\quad\Rightarrow\quad \exists y_2 \in \sigma \text{ s.t. } f(y_2)_2 = (y_2)_2 \\
  &\quad\vdots \\
  L(v_{n-1}) = n-1 &\quad\Rightarrow\quad f(v_{n-1})_{n-1} < (v_{n-1})_{n-1} \\
  L(v_n) = n \neq n-1 &\quad\Rightarrow\quad f(v_n)_{n-1} \geq (v_n)_{n-1} \\
  &\quad\Rightarrow\quad \exists y_{n-1} \in \sigma \text{ s.t. } f(y_{n-1})_{n-1} = (y_{n-1})_{n-1}
  \end{aligned}$$

Given any $x \in \sigma$ and any $\ell \in \{1, \ldots, n-1\}$,

$$
\begin{aligned}
|f(x)_\ell - x_\ell| &\leq |f(x)_\ell - f(y_\ell)_\ell| + |f(y_\ell)_\ell - (y_\ell)_\ell| + |(y_\ell)_\ell - x_\ell| \\
&\leq \frac{\varepsilon}{4n} + 0 + \frac{\varepsilon}{4n} \\
&= \frac{\varepsilon}{2n} \\
|f(x)_n - x_n| &= \left| \left(1 - \sum_{\ell=1}^{n-1} f(x)_\ell\right) - \left(1 - \sum_{\ell=1}^{n-1} x_\ell\right) \right| \\
&= \left| \sum_{\ell=1}^{n-1} (x_\ell - f(x)_\ell) \right| \\
&\leq \sum_{\ell=1}^{n-1} |f(x)_\ell - x_\ell| \\
&\leq (n-1)\frac{\varepsilon}{2n} \\
&< \frac{\varepsilon}{2} \\
\text{So } |f(x) - x| &\leq \|f(x) - x\|_1 \\
&\leq (L-1)\frac{\varepsilon}{2n} + \frac{\varepsilon}{2} \\
&< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\
&= \varepsilon
\end{aligned}
$$

Next we turn to correspondences. We start with the notion of fixed point for a correspondence.

Definition 4 Let X be nonempty and $\Psi : X \to 2^X$ be a correspondence. A point $x^* \in X$ is a *fixed point* of Ψ if $x^* \in \Psi(x^*)$.

Note here that we do *not* require $\Psi(x^*) = \{x^*\}$, that is Ψ need not be single-valued at x^*. So x^* can be a fixed point of Ψ but there may be other elements of $\Psi(x^*)$ different from x^*.

Examples:

1. Let $X = [0, 4]$ and $\Psi : X \to 2^X$ be given by

$$
\Psi(x) = \begin{cases} [x+1, x+2] & \text{if } x < 2 \\ [0, 4] & \text{if } x = 2 \\ [x-2, x-1] & \text{if } x > 2 \end{cases}
$$

   Then $x = 2$ is the unique fixed point of Ψ.

2. Let $X = [0,4]$ and $\Psi : X \to 2^X$ be given by

$$\Psi(x) = \begin{cases} [x+1, x+2] & \text{if } x < 2 \\ [0,1] \cup [3,4] & \text{if } x = 2 \\ [x-2, x-1] & \text{if } x > 2 \end{cases}$$

Then Ψ has no fixed points.

When does a correspondence have a fixed point? Since a function is a special case of a correspondence, it is clear that we can find counterexamples without at least some form of continuity such as upper hemicontinuity, or without compactness and convexity of the domain X. As the examples above illustrate, however some more structure is required (see Figures 8 and 9).

Theorem 5 (Thm. 3.4'. Kakutani's Fixed Point Theorem) *Let $X \subset \mathbf{R}^n$ be a non-empty, compact, convex set and $\Psi : X \to 2^X$ be an upper hemi-continuous correspondence with non-empty, convex, compact values. Then Ψ has a fixed point in X.*

Proof: (sketch) Here, the idea is to use Brouwer's theorem after appropriately approximating the correspondence with a function. The catch is that there won't necessarily exist a continuous selection from Ψ, that is, a continuous function $f : X \to X$ such that $f(x) \in \Psi(x)$ for every $x \in X$. If such a function existed, then by applying Brouwer to f we would have a fixed point of Ψ (because if $\exists x^* \in X$ such that $x^* = f(x^*)$, then $x^* = f(x^*) \in \Psi(x^*)$). Figure 10 gives a simple example of an uhc correspondence Ψ with convex, compact values for which no such continuous selection exists, however.

Instead, we look for a weaker type of approximation. Let $X \subset \mathbf{R}^n$ be a non-empty, compact, convex set, and let $\Psi : X \to 2^X$ be an uhc correspondence with non-empty, compact, convex values. For every $\varepsilon > 0$, define the ε ball about graph Ψ to be

$$B_\varepsilon(\text{ graph } \Psi) = \left\{ z \in X \times X : d(z, \text{ graph } \Psi) = \inf_{(x,y) \in \text{ graph } \Psi} d(z, (x,y)) < \varepsilon \right\}$$

Here d denotes the ordinary Euclidean distance. Since Ψ is a convex-valued correspondence, for every $\varepsilon > 0$ there exists a continuous function $f_\varepsilon : X \to X$ such that graph $f_\varepsilon \subseteq B_\varepsilon(\text{ graph}\Psi)$.[2] See Figure 11 for an illustration of this construction. The ε ball about the graph of Ψ is the area between the dashed lines, which contains the graph of the continuous function f.

Now by letting $\varepsilon \to 0$, this means that we can find a sequence of continuous functions $\{f_n\}$ such that graph $f_n \subseteq B_{\frac{1}{n}}(\text{ graph } \Psi)$ for each n. By Brouwer's Fixed Point Theorem, each function f_n has a fixed point $\hat{x}_n \in X$, and

$$(\hat{x}_n, \hat{x}_n) = (\hat{x}_n, f_n(\hat{x}_n)) \in \text{ graph } f_n \subseteq B_{\frac{1}{n}}(\text{ graph } \Psi) \text{ for each } n$$

[2]This result is Celina's approximation theorem (for example, see Hildenbrand and Kirman, *Equilibrium Analysis* for a reference).

So for each n there exists $(x_n, y_n) \in$ graph Ψ such that

$$d(\hat{x}_n, x_n) < \frac{1}{n} \text{ and } d(\hat{x}_n, y_n) < \frac{1}{n}$$

Since X is compact, $\{\hat{x}_n\}$ has a convergent subsequence $\{\hat{x}_{n_k}\}$, with $\hat{x}_{n_k} \to \hat{x} \in X$. Then $x_{n_k} \to \hat{x}$ and $y_{n_k} \to \hat{x}$. Since Ψ is uhc and closed-valued, it has closed graph, so $(\hat{x}, \hat{x}) \in$ graph Ψ. Thus $\hat{x} \in \Psi(\hat{x})$, that is, $\hat{x}$ is a fixed point of Ψ. ■

Section 6.1(d): Separating Hyperplane Theorems

Convex sets have a number of important geometric and analytic properties. Some of the most important results in convex analysis are those dealing with supporting and separating hyperplanes. There are a number of different results along these lines. We give several here, starting with the most common and widely used version.

Theorem 6 (1.26, Separating Hyperplane Theorem) *Let $A, B \subseteq \mathbf{R}^n$ be nonempty, disjoint convex sets. Then there exists a nonzero vector $p \in \mathbf{R}^n$ such that*

$$p \cdot a \leq p \cdot b \quad \forall a \in A, b \in B$$

We will prove a special case of this result, from which you can derive the general result.

Theorem 7 *Let $Y \subseteq \mathbf{R}^n$ be a nonempty convex set and $x \notin Y$. Then there exists a nonzero vector $p \in \mathbf{R}^n$ such that*

$$p \cdot x \leq p \cdot y \quad \forall y \in Y$$

Proof: We sketch the proof in the special case that Y is compact. We will see that in this case we actually get a stronger conclusion:

$$\exists p \in \mathbf{R}^n,\ p \neq 0 \text{ s.t. } p \cdot x < p \cdot y \quad \forall y \in Y$$

See the following discussion and theorem as well.

Choose $y_0 \in Y$ such that $|y_0 - x| = \inf\{|y - x| : y \in Y\}$; such a point exists because Y is compact, so the distance function $g(y) = |y - x|$ assumes its minimum on Y. Since $x \notin Y$, $x \neq y_0$, so $y_0 - x \neq 0$. Let $p = y_0 - x$. The set

$$H = \{z \in \mathbf{R}^n : p \cdot z = p \cdot y_0\}$$

is the hyperplane perpendicular to p through y_0. See Figure 14. Then

$$\begin{aligned}
p \cdot y_0 &= (y_0 - x) \cdot y_0 \\
&= (y_0 - x) \cdot (y_0 - x + x) \\
&= (y_0 - x) \cdot (y_0 - x) + (y_0 - x) \cdot x \\
&= |y_0 - x|^2 + p \cdot x \\
&> p \cdot x
\end{aligned}$$

We claim that

$$y \in Y \Rightarrow p \cdot y \geq p \cdot y_0$$

If not, suppose there exists $y \in Y$ such that $p \cdot y < p \cdot y_0$. Given $\alpha \in (0,1)$, let

$$w_\alpha = \alpha y + (1-\alpha)y_0$$

Since Y is convex, $w_\alpha \in Y$. Then for α sufficiently close to zero,

$$\begin{aligned}
|x - w_\alpha|^2 &= |x - \alpha y - (1-\alpha)y_0|^2 \\
&= |x - y_0 + \alpha(y_0 - y)|^2 \\
&= |-p + \alpha(y_0 - y)|^2 \\
&= |p|^2 - 2\alpha p \cdot (y_0 - y) + \alpha^2 |y_0 - y|^2 \\
&= |p|^2 + \alpha \left(-2p \cdot (y_0 - y) + \alpha |y_0 - y|^2 \right) \\
&< |p|^2 \quad \text{for } \alpha \text{ close to } 0, \text{ as } p \cdot y_0 > p \cdot y \\
&= |y_0 - x|^2
\end{aligned}$$

Thus for α sufficiently close to zero,

$$|w_\alpha - x| < |y_0 - x|$$

which implies y_0 is not the closest point in Y to x, contradiction. ■

The general version of the Separating Hyperplane Theorem can be derived from this special case by noting that if $A \cap B = \emptyset$, then $0 \notin A - B = \{a - b : a \in A, b \in B\}$.

As we noted in the proof above, in the case of separating a point $\{x\}$ from a set Y we were able to reach a stronger conclusion. In some applications it will be useful to have a stronger result that guarantees this strict separation. Stronger assumptions are required for this, as Figure 15 illustrates. Here is one such result.

Theorem 8 (Strict Separating Hyperplane Theorem) *Let $A, B \subseteq \mathbf{R}^n$ be nonempty, disjoint, closed, convex sets. Then there exists a nonzero vector $p \in \mathbf{R}^n$ such that*

$$p \cdot a < p \cdot b \quad \forall a \in A, b \in B$$

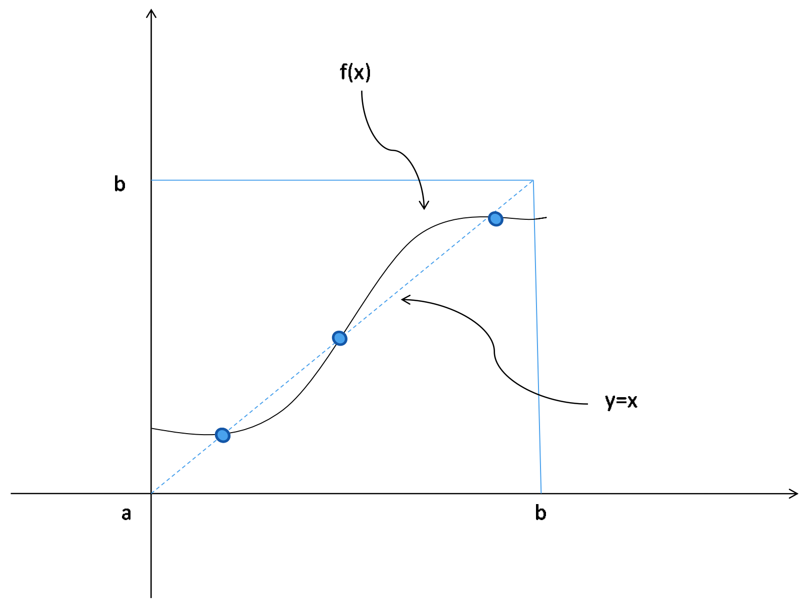

Figure 1: fixed point

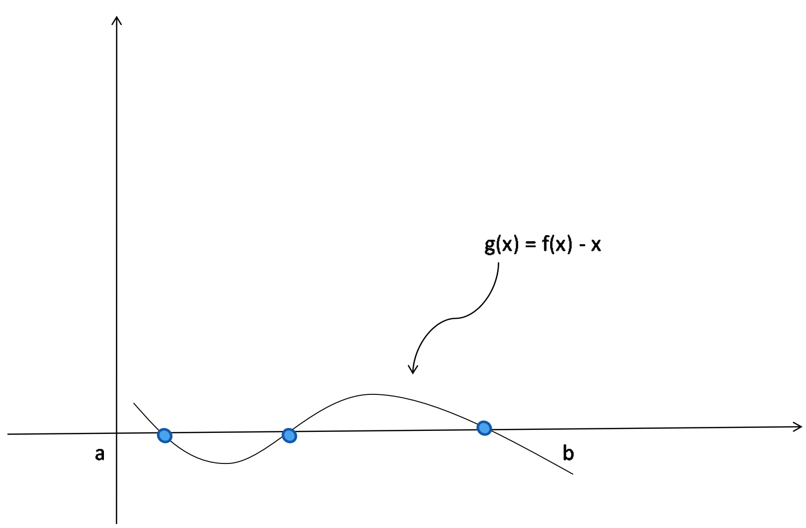

Figure 2: x is a fixed point of f iff $g(x) = f(x) - x = 0$.

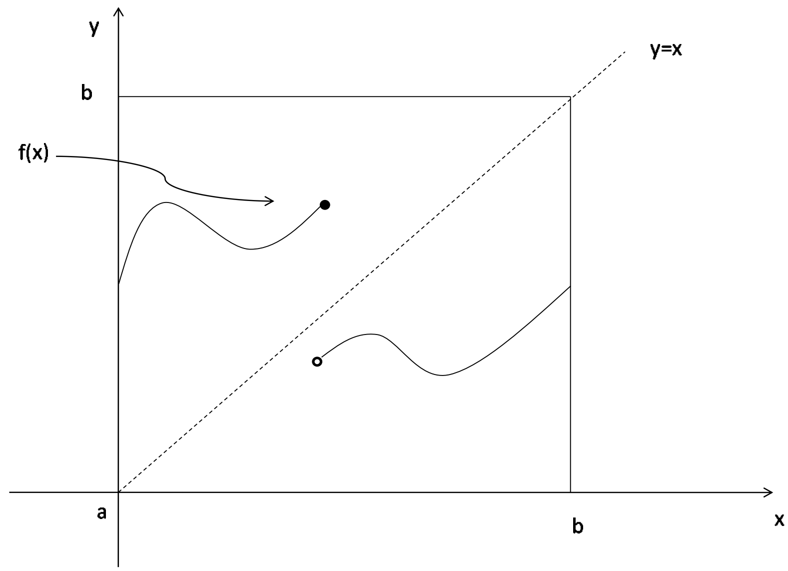

Figure 3: A discontinuous function $f : [a, b] \to [a, b]$ with no fixed point.

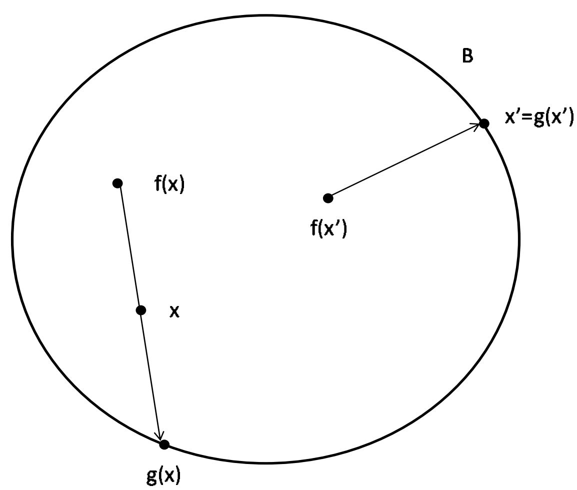

Figure 4: Construction of the function g.

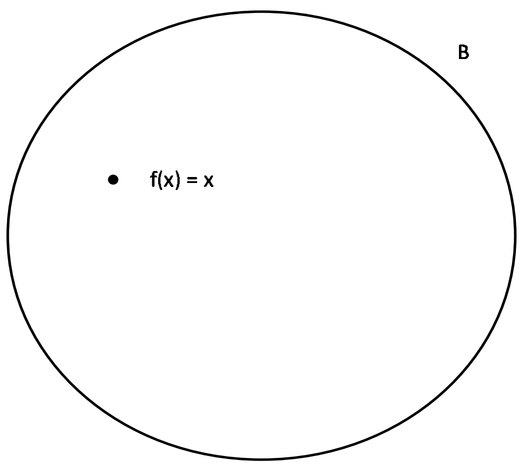

Figure 5: If f has a fixed point x, the construction of $g(x)$ is not possible.

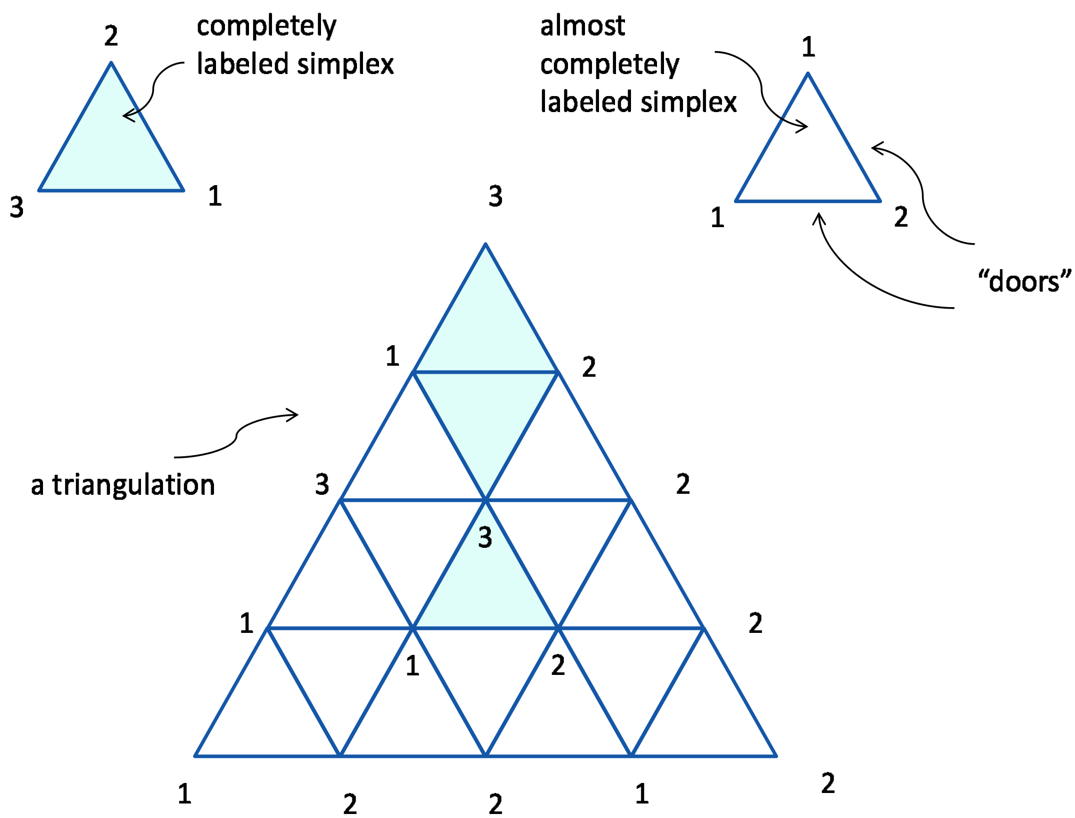

Figure 6: A triangulation, with completely labeled and almost completely labeled simplexes identified.

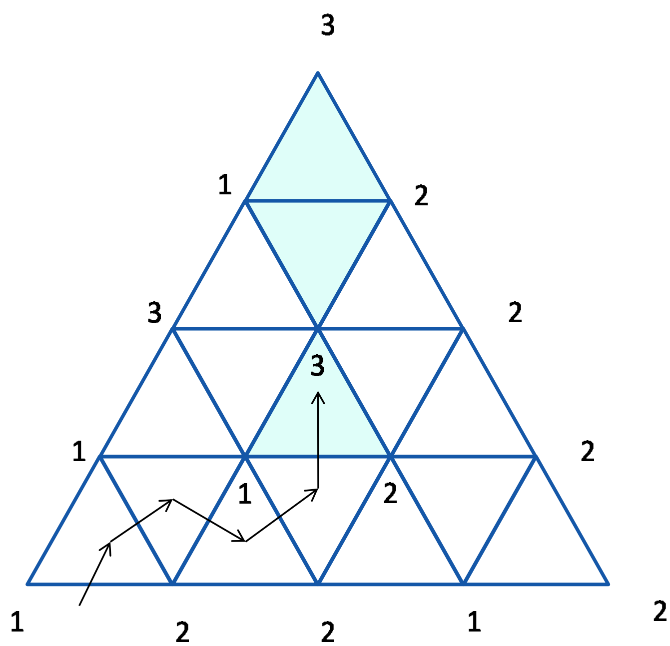

Figure 7: An illustration of Scarf's algorithm to find a completely labeled simplex.

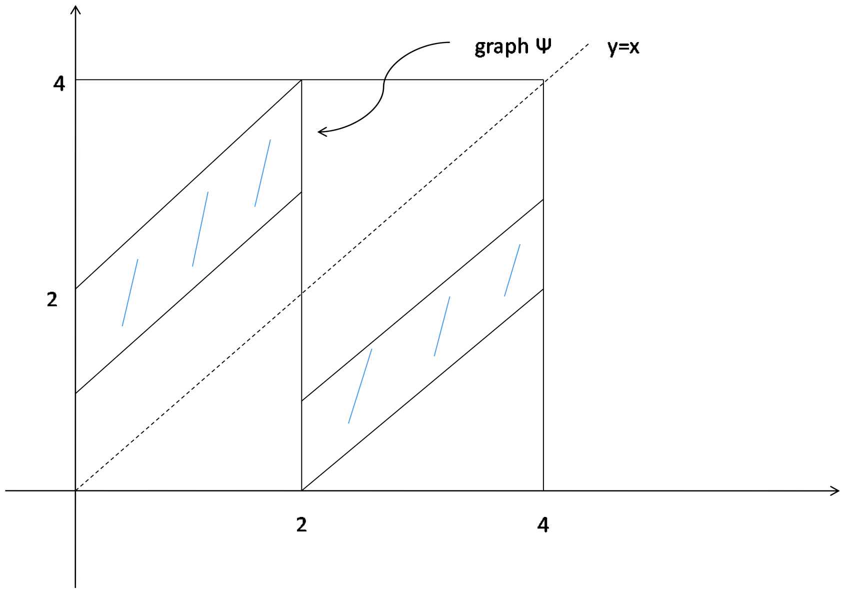

Figure 8: The correspondence Ψ has a fixed point at $x = 2$.

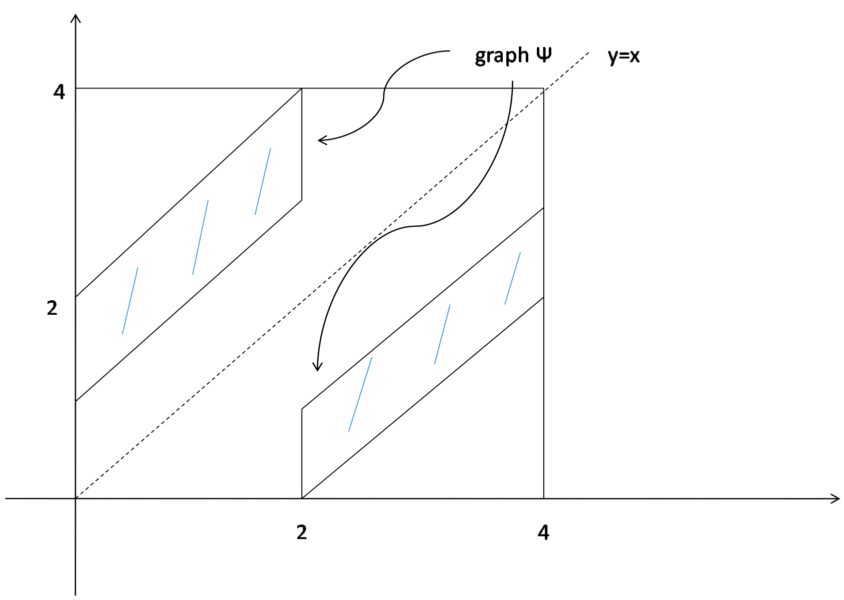

Figure 9: The correspondence Ψ is upper hemicontinuous but has no fixed point on $[0, 4]$.

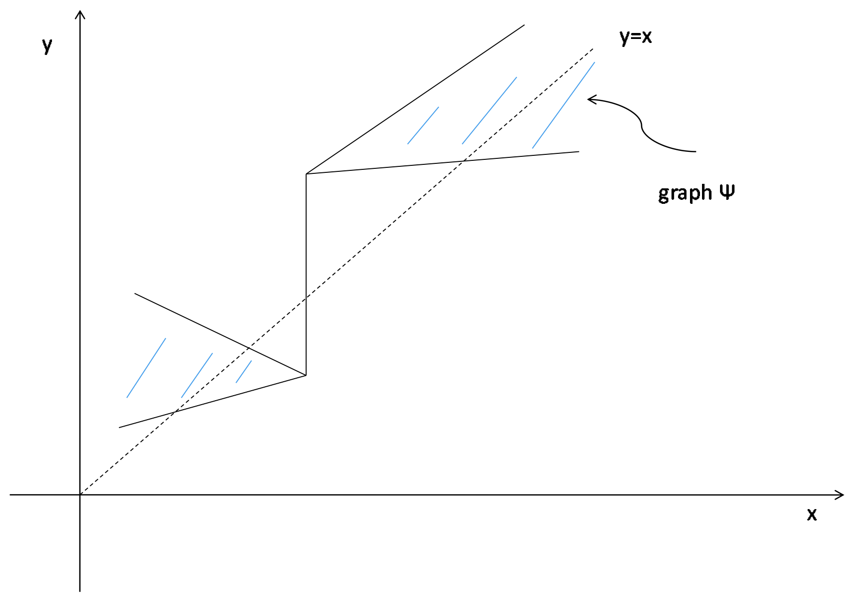

Figure 10: Ψ is uhc with compact, convex values, but has no continuous selection, that is, there is no continuous function f with $f(x) \in \Psi(x)$ for every x.

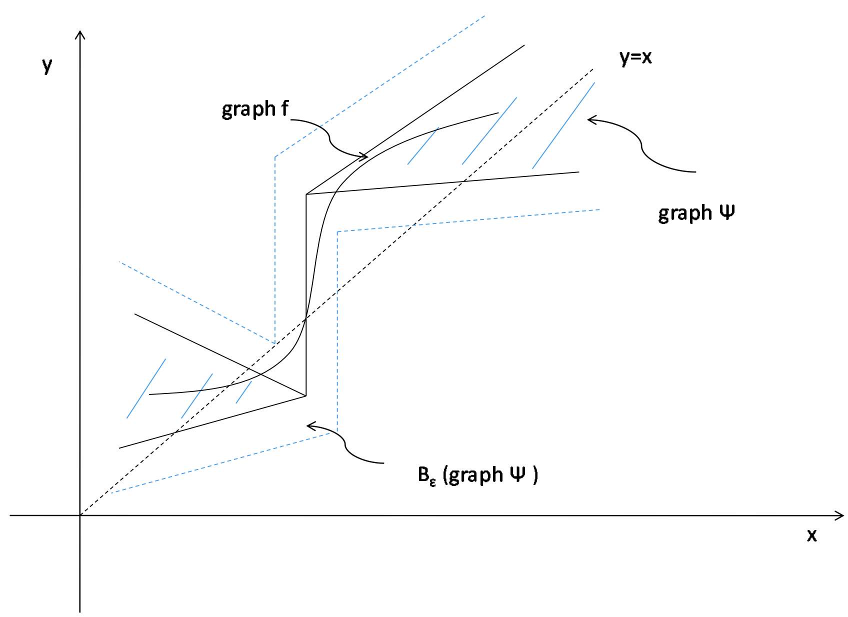

Figure 11: A continuous selection from the ε-ball about the graph of Ψ.

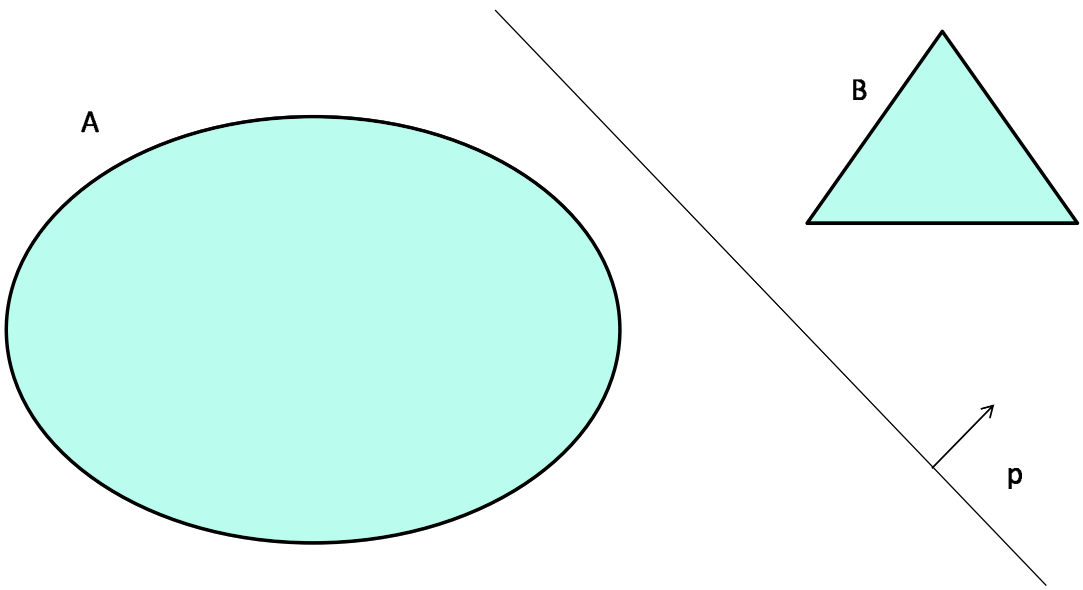

Figure 12: Separating Hyperplane Theorem

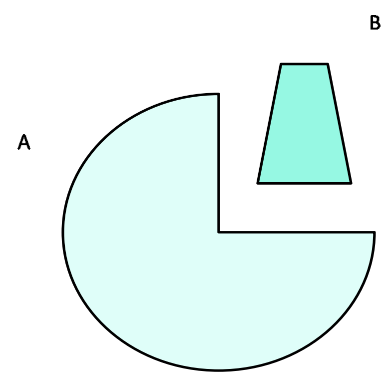

Figure 13: There is no hyperplane separating A and B.

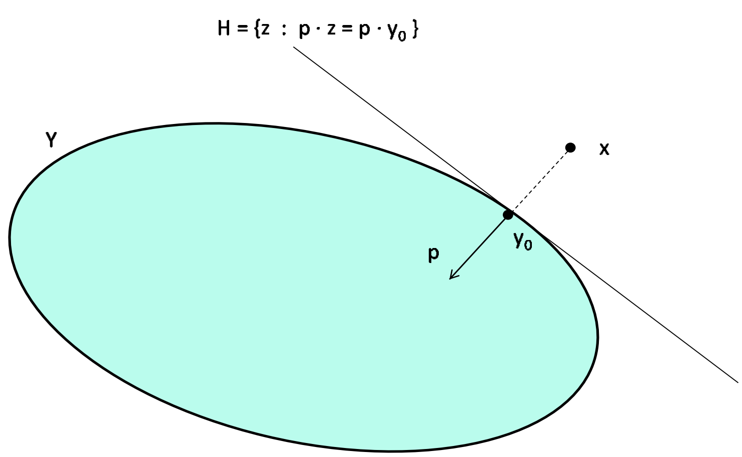

Figure 14: Separating Y from $x \notin Y$.

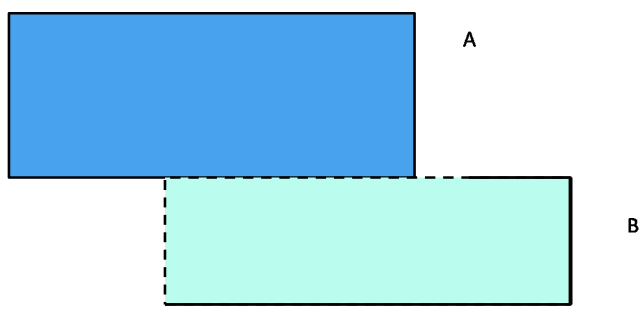

Figure 15: The sets A and B are convex and disjoint, but cannot be strictly separated.