Econ 204 – Problem Set 1

Due Friday August 2 [1]

1. Use induction to prove the following:

   (a) $2^{2n} - 1$ is divisible by 3 for all $n \in \mathbb{N}$.

   (b) $1 + \frac{1}{\sqrt{2}} + \ldots + \frac{1}{\sqrt{n}} \leq 2\sqrt{n}$

2. In the following examples, show that the sets A and B are numerically equivalent by finding a specific bijection between the two.

   (a) $A = [0, 1],\ B = [10, 20]$

   (b) $A = [0, 1],\ B = [0, 1)$

   (c) $A = (-1, 1),\ B = \mathbb{R}$

3. Define the infinite **cartesian product** of a set X with itself as $X^\omega := \prod_{i \in \mathbb{N}} X$. Prove by contradiction that for $X = \{0, 1\}$, X^ω is uncountable. (Hint: suppose there exists a surjective map $f : \mathbb{N} \to X^\omega$, and find an element in X^ω which is not in the image of f).

4. Let X be an arbitrary non-empty subset, and $\mathcal{U} \subset 2^{X \times X}$ be non-empty. For any $E, F \subset X \times X$ define $E^{-1} := \big\{(y, x) : (x, y) \in E\big\}$, and $E \circ F := \big\{(x, z) : (x, y) \in F, (y, z) \in E \text{ for some } y \in X\big\}$. Also call $\Delta := \big\{(x, x) : x \in X\big\}$ the diagonal set. We call $\mathcal{U}$ a *uniformity structure* on X if

   (i) $E \in \mathcal{U}$ implies $\Delta \subset E$.

   (ii) $E, F \in \mathcal{U}$ implies $E \cap F \in \mathcal{U}$.

   (iii) $E \in \mathcal{U}$ implies $F \circ F \subset E$ for some $F \in \mathcal{U}$.

   (iv) $E \in \mathcal{U}$ implies $F^{-1} \subset E$ for some $F \in \mathcal{U}$.

   (v) $E \in \mathcal{U}$ and $E \subset F$ implies $F \in \mathcal{U}$.

   Let $\mathcal{U}$ be a uniformity structure on X. For $x \in X$ and $E \in \mathcal{U}$ define $E[x] := \big\{y \in X : (x, y) \in E\big\}$.

   (a) Show that for every $x \in X$ and every $E \in \mathcal{U}$, $x \in E[x]$.

   (b) Show that $E_1[x] \cap E_2[x] = (E_1 \cap E_2)\,[x]$ for every $x \in X$ and $E_1, E_2 \in \mathcal{U}$.

   (c) Show that $E \in \mathcal{U}$ implies $E^{-1} \in \mathcal{U}$.

   (d) Show that for every $E \in \mathcal{U}$, there is some set $F \in \mathcal{U}$ such that $F \circ F^{-1} \subset E$.

   (e) Show that for every $E \in \mathcal{U}$, there is a *symmetric* set $F \in \mathcal{U}$ such that $F \circ F \subset E$. A set F is called symmetric if $F = F^{-1}$.

   (f) Henceforth assume the family $\mathcal{U}$ is *separating*, namely $\bigcap_{E \in \mathcal{U}} E = \Delta$. Show that for every $x \neq y$, there is some $E \in \mathcal{U}$ such that $(x, y) \notin E$.

   (g) Show that for the above E there exits a symmetric set $F \in \mathcal{U}$ that satisfies $F \circ F \subset E$, and $F[x] \cap F[y] = \emptyset$ for the above $x \neq y$.

[1] In case of any problems with the exercises please email farzad@berkeley.edu

5. Let $\mathcal{U}$ and $\mathcal{Z}$ be two sets, and $P : \mathcal{U} \times \mathcal{Z} \to \mathbb{R}$ be a bounded function. Define the *upper and lower value functions* as:

$$\begin{aligned} V_+ &= \inf_{u \in \mathcal{U}} \sup_{z \in \mathcal{Z}} P(u, z) \\ V_- &= \sup_{z \in \mathcal{Z}} \inf_{u \in \mathcal{U}} P(u, z) \end{aligned} \tag{1}$$

   (a) Show that $V_+ \geq V_-$.

   (b) Call any function $\beta : \mathcal{U} \to \mathcal{Z}$ a *strategy* for the maximizing side. Denote the space of all such strategies as $\mathscr{B}$. Prove the following identity, and explain why it is not in contrast with part (a).

$$V_+ = \inf_{u \in \mathcal{U}} \sup_{\beta \in \mathscr{B}} P(u, \beta(u)) = \sup_{\beta \in \mathscr{B}} \inf_{u \in \mathcal{U}} P(u, \beta(u)) \tag{2}$$

6. Let $f : [a, b] \to \mathbb{R}$. The set $P = \{x_0, x_1, \ldots, x_n\}$ is called a *partition* for $[a, b]$, if $a = x_0 < x_1 < \ldots < x_n = b$. Define $V(f; P) := \sum_{j=1}^{n} \left| f(x_j) - f(x_{j-1}) \right|$. The *variation* of f on $[a, b]$ is defined as

$$V(f; [a, b]) := \sup \left\{ V(f; P) : P \text{ is a partition for } [a, b] \right\}. \tag{3}$$

   When $V(f; [a, b])$ is finite, we say that f is of *bounded variation* on $[a, b]$.

   (a) Show that the class of functions of bounded variation on $[a, b]$ is closed under addition. That is if f and g have bounded variation on $[a, b]$, then $f + g$ also has bounded variation on $[a, b]$.

   (b) Show that if f is of bounded variation on $[a, b]$ and $a \leq c \leq b$, then

$$V(f; [a, b]) = V(f; [a, c]) + V(f; [c, b]). \tag{4}$$