Econ 204 – Problem Set 2

Due Tuesday, August 6

1. Give an example of a complete metric space which is homeomorphic to an incomplete metric space.

2. Let (X, d) be a metric space. Let $\{A_\lambda\}_{\lambda \in \Lambda}$ be a family of connected subsets of X. Assume that $\exists \lambda_0 \in \Lambda$ such that $A_{\lambda_0} \cap A_\lambda \neq \emptyset$ for each $\lambda \in \Lambda$. Show that $S = \bigcup_{\lambda \in \Lambda} A_\lambda$ is a connected set.

3. Let (X, d) be a metric space. Assume $f : X \to \mathbb{R}$ and $g : X \to \mathbb{R}$ are uniformly continuous on (X, d) and $(\mathbb{R}, |\cdot|)$, with $|\cdot|$ the absolute-value norm.

   (a) Show that $f + g : X \to \mathbb{R}$ is uniformly continuous, where $(f + g)(x) = f(x) + g(x)$.

   (b) Show that $\max\{f, g\} : X \to \mathbb{R}$ is uniformly continuous, where $\max\{f, g\}(x) = \max\{f(x), g(x)\}$.

   (c) Give a counterexample to the following statement: $f \cdot g : X \to \mathbb{R}$ *is uniformly continuous on* (X, d) *and* $(\mathbb{R}, |\cdot|)$, where $f \cdot g = f(x) \cdot g(x)$.

4. Take any mapping f from a metric space X into a metric space Y. Prove that f is continuous if and only if $f\left(\overline{A}\right) \subseteq \overline{f(A)}$ for every set A. (Hint: use the closed set characterization of continuity).

5. Prove that a metric space (X, d) is discrete if and only if every function on X into any other metric space (Y, ρ), where Y has at least two distinct elements, is continuous.[1]

6. Suppose T is an operator on a complete metric space (X, d). Prove that the condition

$$d\left(T(x), T(y)\right) < d(x, y) \ \ \forall x, y \in X \ (x \neq y)$$

   does not guarantee the existence of a fixed point of T.

[1] A metric space (X, d) is *discrete* if every subset $A \subset X$ is open. Notice that any set equipped with the discrete metric forms a discrete metric space, but not every discrete metric space necessarily has the discrete metric.