1. Give an example of a complete metric space which is homeomorphic to an incomplete metric space.

2. Let \((X, d)\) be a metric space. Let \(\{A_\lambda\}_{\lambda \in \Lambda}\) be a family of connected subsets of \(X\). Assume that \(\exists \lambda_0 \in \Lambda\) such that \(A_{\lambda_0} \cap A_\lambda \neq \emptyset\) for each \(\lambda \in \Lambda\). Show that \(S = \bigcup_{\lambda \in \Lambda} A_\lambda\) is a connected set.

3. Let \((X, d)\) be a metric space. Assume \(f : X \to \mathbb{R}\) and \(g : X \to \mathbb{R}\) are uniformly continuous on \((X, d)\) and \((\mathbb{R}, |\cdot|)\), with \(|\cdot|\) the absolute-value norm.
   
   (a) Show that \(f + g : X \to \mathbb{R}\) is uniformly continuous, where \((f + g)(x) = f(x) + g(x)\).
   
   (b) Show that \(\max\{f, g\} : X \to \mathbb{R}\) is uniformly continuous, where \(\max\{f, g\}(x) = \max\{f(x), g(x)\}\).
   
   (c) Give a counterexample to the following statement: \(f \cdot g : X \to \mathbb{R}\) is uniformly continuous on \((X, d)\) and \((\mathbb{R}, |\cdot|)\), where \(f \cdot g = f(x) \cdot g(x)\).

4. Take any mapping \(f\) from a metric space \(X\) into a metric space \(Y\). Prove that \(f\) is continuous if and only if \(f(A) \subseteq f(A)\) for every set \(A\). (Hint: use the closed set characterization of continuity).

5. Prove that a metric space \((X, d)\) is discrete if and only if every function on \(X\) into any other metric space \((Y, \rho)\), where \(Y\) has at least two distinct elements, is continuous.\(^1\)

6. Suppose \(T\) is an operator on a complete metric space \((X, d)\). Prove that the condition
   
   \[d(T(x), T(y)) < d(x, y) \quad \forall x, y \in X \quad (x \neq y)\]
   
   does not guarantee the existence of a fixed point of \(T\).

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\(^1\)A metric space \((X, d)\) is **discrete** if every subset \(A \subset X\) is open. Notice that any set equipped with the discrete metric forms a discrete metric space, but not every discrete metric space necessarily has the discrete metric.