Econ 204 – Problem Set 3

Due Friday August 9, 2019 [1]

1. Let (X, d) be a metric space:

   (a) Let $y \in X$ be given. Define the function $d_y : X \to \mathbb{R}$ by

   $$d_y(x) = d(x, y) \tag{1}$$

   Show that d_y is a continuous function on X for each $y \in X$.

   (b) Let A be a subset of X and $x \in X$. Recall that the distance from the point x to the set A is defined as:

   $$\rho(x, A) = \inf \{d(x, a) : a \in A\} \tag{2}$$

   Show that the closure of set A is the set of all points with zero distance to A, that is:

   $$\bar{A} = \{x \in X : \rho(x, A) = 0\} \tag{3}$$

   (c) Now let $A \subset X$ be a compact subset. Show that $\rho(x, A) = d(x, a)$ for some $a \in A$.

2. Let D be the space of all functions $f : [0, 1] \to \mathbb{R}$ such that f is continuous and such that for some $\varepsilon > 0$, $f : (-\varepsilon, 1 + \varepsilon) \to \mathbb{R}$ is differentiable and $f' : (-\varepsilon, 1 + \varepsilon) \to \mathbb{R}$ is continuous. For each $f \in D$, let

   $$\|f\|_\infty = \sup \left\{ |f(t)| : t \in [0, 1] \right\} \quad \text{and} \quad \|f'\|_\infty = \sup \left\{ |f'(t)| : t \in [0, 1] \right\}. \tag{4}$$

   Define the function $\|\cdot\| : D \to \mathbb{R}_+$ by

   $$\|f\| := \|f\|_\infty + \|f'\|_\infty \tag{5}$$

   (a) Show that $(D, \|\cdot\|)$ is a normed vector space.

   (b) Define the function $J : D \to \mathbb{R}$ as

   $$J(f) = \int_0^1 c(x) \left(f'(x)\right)^2 \mathrm{d}x, \tag{6}$$

   where $c : [0, 1] \to \mathbb{R}_+$ is a nonnegative integrable function (its integral over $[0, 1]$ is finite). Prove that J is continuous.

3. Let X be a metric space.

   (a) Let C, D be disjoint open non-empty subsets in X, such that $X = C \cup D$. Suppose $Y \subset X$ is a connected subset. Show Y lies entirely within either C or D. (caution and hint: you can only use the definition of the connected subset presented in lecture 7.)

[1] In case of any problems with the exercises please email farzad@berkeley.edu

(b) Let $\{A_\lambda\}_{\lambda\in\Lambda}$ be a family of connected subsets of X. Assume that $\exists\lambda_0 \in \Lambda$ such that $A_{\lambda_0} \cap A_\lambda \neq \emptyset$ for each $\lambda \in \Lambda$. Show that $S = \bigcup_{\lambda\in\Lambda} A_\lambda$ is a connected set.

4. Define the correspondence $\Gamma : [0,1] \to 2^{[0,1]}$ by:

$$\Gamma(x) = \begin{cases} [0,1] \cap \mathbb{Q} & \text{if } x \in [0,1] \setminus \mathbb{Q} \\ [0,1] \setminus \mathbb{Q} & \text{if } x \in [0,1] \cap \mathbb{Q} \end{cases}. \tag{7}$$

Show that Γ is not continuous, but it is lower-hemicontinuous. Is Γ upper-hemicontinuous at any rational? At any irrational? Does this correspondence have a closed graph?

5. Let X be a metric space, and $I : X \to \mathbb{R}_+$ be a lower semi-continuous function [2].

(a) Prove that for every given $\varepsilon > 0$ there exists an open set U_ε containing $x \in X$ such that

$$\inf\{I(y) : y \in U_\varepsilon\} \geq I(x) - \varepsilon. \tag{8}$$

(b) Let $x \in X$. For each $n \in \mathbb{N}$ let

$$m_n = \inf\left\{I(y) : y \in B_{1/n}(x)\right\}. \tag{9}$$

Show that $\{m_n\}$ is an increasing sequence and that $m_n \to I(x)$.

6. Let $\mathcal{K}$ be the collection of all non-empty closed subsets of $[0,1] \times [0,1]$. For $A \in \mathcal{K}$ and $\delta > 0$, let A_δ denote the union of all closed disk of radius δ centered at points of A. Specifically, $A_\delta = \{y \in [0,1] \times [0,1] : d(y,a) \leq \delta \text{ for some } a \in A\}$. Define $\mu : \mathcal{K} \times \mathcal{K} \to \mathbb{R}_+$ by

$$\mu(A,B) = \inf\{\delta > 0 : A \subset B_\delta, B \subset A_\delta\}. \tag{10}$$

(a) Prove $(\mathcal{K}, \mu)$ is a metric space.

(b) Let $\{B_n\} \subset \mathcal{K}$ be a decreasing sequence, i.e $B_n \supset B_{n+1}$ for $n \in \mathbb{N}$. Define $B = \cap_{n\in\mathbb{N}} B_n$, and prove $B_n \to B$ in $\mathcal{K}$.

(c) Show that the metric space $(\mathcal{K}, \mu)$ is complete (Hint: use the previous part).

[2] A function $I : X \to \mathbb{R}$ is called lower semi-continuous *iff* for every α the set $\{x : I(x) > \alpha\}$ is open in X.