Econ 204 – Problem Set 3
Due Friday August 9, 2019

1. Let \((X,d)\) be a metric space:

(a) Let \(y \in X\) be given. Define the function \(d_y : X \to \mathbb{R}\) by

\[
d_y(x) = d(x,y)
\]  

(1)

Show that \(d_y\) is a continuous function on \(X\) for each \(y \in X\).

(b) Let \(A\) be a subset of \(X\) and \(x \in X\). Recall that the distance from the point \(x\) to the set \(A\) is defined as:

\[
\rho(x,A) = \inf \{d(x,a) : a \in A\}
\]  

(2)

Show that the closure of set \(A\) is the set of all points with zero distance to \(A\), that is:

\[
\bar{A} = \{x \in X : \rho(x,A) = 0\}
\]  

(3)

(c) Now let \(A \subset X\) be a compact subset. Show that \(\rho(x,A) = d(x,a)\) for some \(a \in A\).

2. Let \(D\) be the space of all functions \(f : [0,1] \to \mathbb{R}\) such that \(f\) is continuous and such that for some \(\varepsilon > 0\), \(f : (-\varepsilon,1+\varepsilon) \to \mathbb{R}\) is differentiable and \(f' : (-\varepsilon,1+\varepsilon) \to \mathbb{R}\) is continuous. For each \(f \in D\), let

\[
\|f\|_\infty = \sup \{ |f(t)| : t \in [0,1] \} \quad \text{and} \quad \|f'\|_\infty = \sup \{ |f'(t)| : t \in [0,1] \}.
\]  

(4)

Define the function \(\|\cdot\| : D \to \mathbb{R}_+\) by

\[
\|f\| := \|f\|_\infty + \|f'\|_\infty
\]  

(5)

(a) Show that \((D,\|\cdot\|)\) is a normed vector space.

(b) Define the function \(J : D \to \mathbb{R}\) as

\[
J(f) = \int_0^1 c(x) \left(f'(x)\right)^2 \, dx,
\]  

(6)

where \(c : [0,1] \to \mathbb{R}_+\) is a nonnegative integrable function (its integral over \([0,1]\) is finite). Prove that \(J\) is continuous.

3. Let \(X\) be a metric space.

(a) Let \(C, D\) be disjoint open non-empty subsets in \(X\), such that \(X = C \cup D\). Suppose \(Y \subset X\) is a connected subset. Show \(Y\) lies entirely within either \(C\) or \(D\). (caution and hint: you can only use the definition of the connected subset presented in lecture 7.)

1In case of any problems with the exercises please email farzad@berkeley.edu
(b) Let \( \{A_\lambda\}_{\lambda \in \Lambda} \) be a family of connected subsets of \( X \). Assume that \( \exists \lambda_0 \in \Lambda \) such that \( A_{\lambda_0} \cap A_\lambda \neq \emptyset \) for each \( \lambda \in \Lambda \). Show that \( S = \bigcup_{\lambda \in \Lambda} A_\lambda \) is a connected set.

4. Define the correspondence \( \Gamma : [0,1] \rightarrow 2^{[0,1]} \) by:

\[
\Gamma(x) = \begin{cases} 
[0,1] \cap \mathbb{Q} & \text{if } x \in [0,1] \setminus \mathbb{Q} \\
[0,1] \setminus \mathbb{Q} & \text{if } x \in [0,1] \cap \mathbb{Q}
\end{cases}
\]  

(7)

Show that \( \Gamma \) is not continuous, but it is lower-hemicontinuous at any rational? At any irrational? Does this correspondence have a closed graph?

5. Let \( X \) be a metric space, and \( I : X \rightarrow \mathbb{R}_+ \) be a lower semi-continuous function \(^2\).

(a) Prove that for every given \( \varepsilon > 0 \) there exists an open set \( U_\varepsilon \) containing \( x \in X \) such that

\[
\inf \{ I(y) : y \in U_\varepsilon \} \geq I(x) - \varepsilon.
\]  

(8)

(b) Let \( x \in X \). For each \( n \in \mathbb{N} \) let

\[
m_n = \inf \{ I(y) : y \in B_{1/n}(x) \}.
\]  

(9)

Show that \( \{m_n\} \) is an increasing sequence and that \( m_n \rightarrow I(x) \).

6. Let \( \mathcal{K} \) be the collection of all non-empty closed subsets of \([0,1] \times [0,1] \). For \( A \in \mathcal{K} \) and \( \delta > 0 \), let \( A_\delta \) denote the union of all closed disk of radius \( \delta \) centered at points of \( A \). Specifically, \( A_\delta = \{ y \in [0,1] \times [0,1] : d(y,a) \leq \delta \text{ for some } a \in A \} \). Define \( \mu : \mathcal{K} \times \mathcal{K} \rightarrow \mathbb{R}_+ \) by

\[
\mu(A, B) = \inf \{ \delta > 0 : A \subset B_\delta, B \subset A_\delta \}.
\]  

(10)

(a) Prove \( (\mathcal{K}, \mu) \) is a metric space.

(b) Let \( \{B_n\} \subset \mathcal{K} \) be a decreasing sequence, i.e \( B_n \supset B_{n+1} \) for \( n \in \mathbb{N} \). Define \( B = \cap_{n \in \mathbb{N}} B_n \), and prove \( B_n \rightarrow B \) in \( \mathcal{K} \).

(c) Show that the metric space \( (\mathcal{K}, \mu) \) is complete (Hint: use the previous part).

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\(^2\) A function \( I : X \rightarrow \mathbb{R} \) is called lower semi-continuous iff for every \( \alpha \) the set \( \{x : I(x) > \alpha \} \) is open in \( X \).