Econ 204 – Problem Set 4

Due Tuesday, August 13

1. Let A be an $n \times n$ matrix.
   (a) Show that if λ is an eigenvalue of A, then λ^k is an eigenvalue of A^k for $k \in \mathbb{N}$.
   (b) Show that if λ is an eigenvalue of the matrix A and A is invertible, then $1/\lambda$ is an eigenvalue of A^{-1}.
   (c) Find an expression for $\det(A)$ in terms of the eigenvalues of A.
   (d) The *eigenspace* of an eigenvalue λ_i is the kernel of $A - \lambda_i I$. Show that the eigenspace of any matrix A belonging to an eigenvalue λ_i is a vector space.

2. Let V be an n-dimensional vector space. Call a linear operator $T : V \to V$ *idempotent* if $T \circ T = T$. Prove that all such operators are diagonalizable (that is, any matrix representation $A = Mtx_U(T)$ is diagonalizable). What are the eigenvalues?

3. Let V be a finite-dimensional vector space and $W \subset V$ be a vector subspace. Prove that W has a complement in V, i.e., there exists a vector subspace $W' \subset V$ such that $W \cap W' = \{0\}$ and $W + W' = V$.

4. Let U and V be vector spaces. Suppose $T : U \to V$ is a linear transformation and $v \in V$. Prove that, if the preimage $T^{-1}(v)$ is non-empty, and $u \in T^{-1}(v)$, then $T^{-1}(v) = \{u + z | z \in \ker T\} = u + \ker T$.

5. Call a function $T : V \to W$ *additive* if $T(x + y) = T(x) + T(y)$ for every $x, y \in V$. Prove the following:
   (a) Any rational additive function $T : \mathbb{Q} \to \mathbb{Q}$ is linear.
   (b) There is some real additive function $T : \mathbb{R} \to \mathbb{R}$ that is not linear.
   (c) If an additive function $T : \mathbb{R} \to \mathbb{R}$ is nonlinear, then its graph $\Gamma = \{(x, T(x)) : x \in \mathbb{R}\}$ is dense in $\mathbb{R}^2$.
   (d) If an additive function $T : \mathbb{R} \to \mathbb{R}$ is continuous at a point x_0, then it is linear.

6. (a) Show that $\det : \mathbb{R}^{n \times n} \to \mathbb{R}$ is continuous.
   (b) Use the continuity of the determinant to prove that the set of all invertible matrices is an open, dense subset of all square matrices.