Econ 204 – Problem Set 6

Due Monday, August 19 in Dami's mailbox (Evans 612)

1. Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a C^1 function. Define $F : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ by

$$F(x, \omega) = f(x) + \omega$$

   Show that there is a set $\Omega_0 \subset \mathbb{R}^n$ of Lebesgue measure zero such that, if $\omega \notin \Omega_0$, then for each x_0 satisfying $F(x_0, \omega_0) = 0$ there is an open set U containing x_0, an open set V containing ω_0, and a C^1 function $h : V \to U$ such that for all $\omega \in V$, $x = h(\omega)$ is the unique element of U satisfying $F(x, \omega) = 0$.

2. Let $A \in \mathbb{R}^{m \times n}$ and $b \in \mathbb{R}^{m \times 1}$. Then, show that exactly one of the following two conditions hold:

   - $\exists x \in \mathbb{R}^n$ such that $Ax = b$, with $x \geq 0$;
   - $\exists y \in \mathbb{R}^{1 \times m}$ such that $A'y \geq 0$, and $y'b < 0$.

   Hint: you may want to use the following definition and its properties. If $v_1, v_2, ..., v_n$ are the columns of A, define

$$Q = \text{cone}(A) \equiv \left\{ s \in \mathbb{R}^m : s = \sum_{i=1}^{n} \lambda_i v_i, \lambda_i \geq 0, \forall i \right\},$$

   i.e., Q is the set of all conic combinations of the columns of A. Note that Q is non-empty ($0 \in Q$), and assume it is closed and convex (you should be able to prove this!).

3. Call a vector $\pi \in \mathbb{R}^n$ a *probability vector* if

$$\sum_{i}^{n} \pi_i = 1 \text{ and } \pi_i \geq 0 \ \forall i$$

   We say there are n states of the world, and π_i is the probability that state i occurs. Suppose there are two traders (trader 1 and trader 2) who each have a set of prior probability distributions (Π_1 and Π_2) which are nonempty, convex, and compact. Call a *trade* a vector $f \in \mathbb{R}^n$, which denotes the net transfer trader 1 receives in each state of the world (and thus $-f$ is the net transfer trader 2 receives in each state of the world). A trade is *agreeable* if

$$\inf_{\pi \in \Pi_1} \sum_{i=1}^{n} \pi_i f_i > 0 \quad \text{and} \quad \inf_{\pi \in \Pi_2} \sum_{i=1}^{n} \pi_i(-f_i) > 0$$

   Prove that there exists an agreeable trade if and only if there is no common prior (that is, $\Pi_1 \cap \Pi_2 = \varnothing$).

4. Prove the following for any convex set $A \subset \mathbb{R}^n$:

   (a) $\text{int}\, A$ and $\overline{A}$ are convex.

   (b) If A is closed, then A has a unique smallest element (in terms of norm).

(c) A is connected.

5. a) *Berge's Maximum Theorem:* Let $X \subset \mathbb{R}^n$ and $Y \subset \mathbb{R}^m$. Consider the function $f : X \times Y \to \mathbb{R}$ and the correspondence $\Gamma : Y \twoheadrightarrow X$. Define $v(y) = \max_{x \in \Gamma(y)} f(x, y)$ and $\Omega(y) = \arg\max_{x \in \Gamma(y)} f(x, y)$. Suppose f and Γ are continuous, and that Γ has non-empty compact values. Show that v is continuous and Ω is uhc with non-empty compact values. *Hint*: you may find useful to use the sequential definitions of uhc and lhc.

   b) Assume that Γ also has convex values. Show that if f is quasi-concave in x, Ω has convex values.[1]

   c) Let $\mathcal{S}(I, (u^i, S^i, \Gamma^i)_{i \in I})$ denote a *social game*, where I is the (finite) set of players, and $u^i : \prod_{j \in I} S^j \to \mathbb{R}$ is the objective function of player $i \in I$ defined over $s = \left(s^j; j \in I\right) \in \prod_{j \in I} S^j$, with $S^j \subset \mathbb{R}^{n_j}$, $n_j > 0$. Each player i chooses $s^i \in \arg\max_{s \in \Gamma^i(s_{-i})} u^i(s, s_{-i})$, with $s_{-i} := (s_j; j \in I \setminus \{i\})$, and $\Gamma^i(s_{-i}) \subset S^i$. Define an equilibrium for the social game $\mathcal{S}(I, (u^i, S^i, \Gamma^i)_{i \in I})$ as a vector $\bar{s} = \left(\bar{s}^i; i \in I\right)$ such that, $\forall i \in I$, $u^i(\bar{s}) \geq u^i(s, \bar{s}_{-i})$, $\forall s \in \Gamma^i\left(\bar{s}_{-i}\right)$, where $\bar{s}_{-i} := \left(\bar{s}^j; j \neq i\right)$.

   Assume S^i is convex, compact, and non-empty for each $i \in I$, and that u^i is continuous and quasi-concave in s^i for each $i \in I$. Use the previous parts of this question to show that, if $\{\Gamma^i\}_{i \in I}$ are continuous and have compact, convex, and non-empty values, then an equilibrium for $\mathcal{S}(I, (u^i, S^i, \Gamma^i)_{i \in I})$ exists.

6. Consider the following inhomogeneous second-order linear differential equation:

$$x''(t) - 2x'(t) + x(t) = \sin(t)$$

   (a) Find the general solution of the corresponding homogeneous equation.

   (b) Find a particular solution of the original inhomogeneous equation satisfying the initial conditions $x(0) = 1$ and $x'(0) = 0$.

   (c) Find the general solution of the original inhomogeneous equation.

[1] A function $f : X \to \mathbb{R}$ is quasi-concave if for all $x_1, x_2 \in X$ and $\lambda \in [0, 1]$, $f(\lambda x_1 + (1-\lambda) x_2) \leq \max\{f(x_1), f(x_2)\}$.