Economics 204 Summer/Fall 2019
Final Exam – Suggested Solutions

Answer all of the questions below. Be as complete, correct, and concise as possible. There are 6 questions for a total of 165 points possible; point values for each problem are in parentheses. For questions with subparts, each subpart is worth the same number of points. You have 180 minutes to complete the exam. Use the points as a guide to allocating your time. You may use any result from class with appropriate references unless you are specifically being asked to prove it.

1. (15) Define or state each of the following.
   (a) eigenvalue of a linear transformation $T : X \to Y$ between vector spaces X and Y over the same field F
   (b) Cauchy sequence in a metric space (X, d)
   (c) Separating Hyperplane Theorem

   Solution: See notes.

2. (30) Show that for every $n \in \mathbb{N} = \{1, 2, 3, \ldots\}$,

$$\sum_{k=1}^{n} (2k-1) = n^2$$

Solution: The proof below is by induction. For the base case, let $n = 1$. Then

$$\sum_{k=1}^{1} (2k-1) = 2 - 1 = 1 = 1^2$$

So the claim is true for $n = 1$.

For the induction hypothesis, suppose for some $n \geq 1$

$$\sum_{k=1}^{n} (2k-1) = n^2$$

Then consider $n + 1$.

$$\begin{aligned}
\sum_{k=1}^{n+1} (2k-1) &= \sum_{k=1}^{n} (2k-1) + (2(n+1) - 1) \\
&= n^2 + 2n + 2 - 1 \quad \text{by the induction hypothesis} \\
&= n^2 + 2n + 1 = (n+1)^2
\end{aligned}$$

Thus the claim is true for $n + 1$. Thus by induction the claim is true for all $n \in \mathbb{N}$.

3. (30) Let X and Y be vector spaces over the same field F, and let $T : X \to Y$ be a linear transformation.

(a.) Show that $\ker T$ is a vector subspace of X, and $\operatorname{Im} T$ is a vector subspace of Y.

Solution: To show that $\ker T$ is a vector subspace of X, let $x, y \in \ker T$ and $\alpha, \beta \in F$. Then

$$T(\alpha x + \beta y) = \alpha T(x) + \beta T(y) = \alpha 0 + \beta 0 = 0$$

where the first equality follows from linearity of T and the second from the assumption that $x, y \in \ker T$, so $T(x) = T(y) = 0$ by definition. Thus $\alpha x + \beta y \in \ker T$. Thus $\ker T$ is a vector subspace of X.

Similarly, to show that $\operatorname{Im} T$ is a vector subspace of Y, let $v, w \in \operatorname{Im} T$ and $\alpha, \beta \in F$. Then by definition, there exist $x, y \in X$ such that $T(x) = v$ and $T(y) = w$. Then $\alpha x + \beta y \in X$ because X is a vector space, and

$$T(\alpha x + \beta y) = \alpha T(x) + \beta T(y) = \alpha v + \beta w$$

where the first equality follows from linearity of T and the second by definition of x and y. Thus $\alpha v + \beta w \in \operatorname{Im} T$. Thus $\operatorname{Im} T$ is a vector subspace of Y.

(b.) Suppose X is finite-dimensional. Show that $\dim X = \dim \ker T + \operatorname{Rank} T$.

Solution: Let $n = \dim X$. By definition, $\operatorname{Rank} T = \dim \operatorname{Im} T$. By (a), $\ker T$ is a vector subspace of X. Let $V = \{v_1, \ldots, v_r\}$ be a basis for $\ker T$. Note that V must be finite and have cardinality at most n, since $V \subseteq X$ is linearly independent. Then extend V to a basis $\{v_1, \ldots, v_r\} \cup \{w_1, \ldots, w_k\}$ for X. By definition, $r + k = n$, and $r = \dim \ker T$.

Now consider $U = \{T(w_1), \ldots, T(w_k)\}$. Claim U is a basis for $\operatorname{Im} T$. To see that U is linearly independent, suppose

$$\sum_{i=1}^{k} \alpha_i T(w_i) = 0 \quad \text{for some } \alpha_1, \ldots, \alpha_k \in F$$

Then since T is linear,

$$\begin{aligned}
& 0 = \sum_{i=1}^{k} \alpha_i T(w_i) = T(\sum_{i=1}^{k} \alpha_i w_i) \\
\Rightarrow \quad & \sum_{i=1}^{k} \alpha_i w_i \in \ker T \\
\Rightarrow \quad & \sum_{i=1}^{k} \alpha_i w_i = \sum_{j=1}^{r} \beta_j v_j \quad \text{for some } \beta_1, \ldots, \beta_r \in F \\
\Rightarrow \quad & \sum_{i=1}^{k} \alpha_i w_i - \sum_{j=1}^{r} \beta_j v_j = 0
\end{aligned}$$

But $\{v_1, \ldots, v_r\} \cup \{w_1, \ldots, w_k\}$ is linearly independent, so $\alpha_i = \beta_j = 0$ for all i, j. Thus $\{T(w_1), \ldots, T(w_k)\}$ is linearly independent. Now claim $\{T(w_1), \ldots, T(w_k)\}$ spans $\operatorname{Im} T$. To see this, let $y \in \operatorname{Im} T$. Then there exists $x \in X$ such that $T(x) = y$. Since $x \in X$, there exist $\alpha_1, \ldots, \alpha_k, \beta_1, \ldots, \beta_r \in F$ such that

$$x = \sum_{i=1}^{k} \alpha_i w_i + \sum_{j=1}^{r} \beta_j v_j$$

Then

$$\begin{aligned} T(x) &= T(\sum_{i=1}^{k} \alpha_i w_i + \sum_{j=1}^{r} \beta_j v_j) \\ &= \sum_{i=1}^{k} \alpha_i T(w_i) + \sum_{j=1}^{r} \beta_j T(v_j) \quad \text{since } T \text{ is linear} \\ &= \sum_{i=1}^{k} \alpha_i T(w_i) \quad \text{since } T(v_j) = 0 \ \forall j \end{aligned}$$

So $y = T(x) = \sum_{i=1}^{k} \alpha_i T(w_i)$. Thus $\operatorname{Im} T \subseteq \operatorname{span}\{T(w_1), \ldots, T(w_k)\}$. Given $\alpha_1, \ldots, \alpha_k \in F$, $\sum_{i=1}^{k} \alpha_i w_i \in X$ and $\sum_{i=1}^{k} \alpha_i T(w_i) = T(\sum_{i=1}^{k} \alpha_i w_i) \in \operatorname{Im} T$. Thus $\operatorname{span}\{T(w_1), \ldots, T(w_k)\} \subseteq \operatorname{Im} T$, which establishes that $\operatorname{Im} T = \operatorname{span}\{T(w_1), \ldots, T(w_k)\}$. Then by definition, $\operatorname{Rank} T = \dim \operatorname{Im} T = |U| = k$. So

$$\dim X = n = r + k = \dim \ker T + \operatorname{Rank} T$$

4. (30) Let $X \subseteq \mathbb{R}^n$ be open and $f : X \to \mathbb{R}$ be differentiable on X. Suppose $x^* \in X$ and $f(x^*) \geq f(x)$ for all $x \in X$. Show that $Df(x^*) = 0$.

Solution: Since f is differentiable on X and $x^* \in X$,

$$Df(x^*) = (\frac{\partial f}{\partial x_1}(x^*), \ldots, \frac{\partial f}{\partial x_n}(x^*)$$

Then it suffices to show that $\frac{\partial f}{\partial x_i}(x^*) = 0$ for each $i = 1, \ldots, n$. To that end, note that by definition

$$\frac{\partial f}{\partial x_i}(x^*) = \lim_{h \to 0} \frac{f(x^* + he_i) - f(x^*)}{h}$$

where $e_i = (0, \ldots 1, \ldots, 0$ is the i^{th} standard basis vector in $\mathbb{R}^n$ and $h \in \mathbb{R}$, $h \neq 0$. Then for all $h \in \mathbb{R}$, $f(x^*) \geq f(x^* + he_i)$, so

$$f(x^* + he_i) - f(x^*) \leq 0$$

Now consider a sequence $h_n \to 0$ such that $h_n > 0$ for all n (e.g., $h_n = \frac{1}{n}$ for each n). From above,

$$\frac{\partial f}{\partial x_i}(x^*) = \lim_{n \to \infty} \frac{f(x^* + h_n e_i) - f(x^*)}{h_n}$$

Since

$$\frac{f(x^* + h_n e_i) - f(x^*)}{h_n} \leq 0 \quad \forall n$$

this implies

$$\frac{\partial f}{\partial x_i}(x^*) \leq 0$$

Similarly, consider a sequence $h_n \to 0$ such that $h_n < 0$ for all n (e.g., $h_n = -\frac{1}{n}$ for each n). From above, since

$$\frac{f(x^* + h_n e_i) - f(x^*)}{h_n} \geq 0 \quad \forall n$$

this implies

$$\frac{\partial f}{\partial x_i}(x^*) \geq 0$$

Thus $\frac{\partial f}{\partial x_i}(x^*) = 0$.

5. (30) Let (X, d) be a metric space and $C \subseteq X$ be compact. Let $U \subseteq X$ be an open set such that $C \subseteq U$. Show that there exists $\varepsilon > 0$ such that

$$B_\varepsilon(C) = \bigcup_{x \in C} B_\varepsilon(x) \subseteq U$$

Solution: Since $C \subseteq U$ and U is open, for each $x \in C$ there exists $\varepsilon_x > 0$ such that $B_{\varepsilon_x}(x) \subseteq U$. Then $\{B_{\frac{\varepsilon_x}{2}}(x) : x \in C\}$ is an open cover of C. Since C is compact, there exists $x_1, \ldots, x_n$ such that

$$C \subseteq B_{\frac{\varepsilon_{x_1}}{2}}(x_1) \cup \cdots \cup B_{\frac{\varepsilon_{x_n}}{2}}(x_n)$$

Now let $\varepsilon = \min\{\frac{\varepsilon_{x_1}}{2}, \ldots, \frac{\varepsilon_{x_n}}{2}\} > 0$. Let $y \in B_\varepsilon(C)$, so there exists $x \in C$ such that $y \in B_\varepsilon(x)$. Since $x \in C$, there exists i such that $x \in B_{\frac{\varepsilon_{x_i}}{2}}(x_i)$. Then

$$d(y, x_i) \leq d(y, x) + d(x, x_i) < \varepsilon + \frac{\varepsilon_{x_i}}{2} \leq \frac{\varepsilon_{x_i}}{2} + \frac{\varepsilon_{x_i}}{2} = \varepsilon_{x_i}$$

So $y \in B_{\varepsilon_{x_i}}(x_i) \subseteq U$. Thus $B_\varepsilon(C) \subseteq U$.

Here is an argument using sequential compactness instead. Suppose by way of contradiction that $\nexists \varepsilon > 0$ such that $B_\varepsilon(C) \subseteq U$. Then for each n there exists $y_n \in B_{\frac{1}{n}}(C)$ such that $y_n \notin U$. Since $y_n \in B_{\frac{1}{n}}(C)$, there exists $x_n \in C$ such that $d(y_n, x_n) < \frac{1}{n}$. Then $\{x_n\} \subseteq C$ and C is compact, hence sequentially compact, so there is a convergent subsequence $x_{n_k} \to x \in C$. Since $d(y_n, x_n) < \frac{1}{n}$ for each n, the subsequence $\{y_{n_k}\}$ also converges, and $y_{n_k} \to x$. But $x \in C \subseteq U$ and U is open, so there exists $\delta > 0$ such that $B_\delta(x) \subseteq U$. Since $y_{n_k} \notin U$ for each n_k, this implies $y_{n_k} \not\to x$, a contradiction. Thus there exists $\varepsilon > 0$ such that $B_\varepsilon(C) \subseteq U$.

6. (30) Let $a, b \in \mathbb{R}$ with $a \leq b$.

a. Let $\Psi : [a, b] \rightarrow 2^{\mathbb{R}}$ be a correspondence that is continuous and nonempty-valued, so $\Psi(x) \subseteq \mathbb{R}$ is a nonempty set for each $x \in [a, b]$. Suppose $y \geq 0\ \forall\, y \in \Psi(a)$, and $y \leq 0\ \forall\, y \in \Psi(b)$ (so $\Psi(a) \subseteq [0, \infty)$ and $\Psi(b) \subseteq (-\infty, 0]$). Show that there exists $c \in [a, b]$ such that $0 \in \Psi(c)$.

Solution: If $0 \in \Psi(a)$ or $0 \in \Psi(b)$ then we are done. So without loss of generality suppose $y > 0$ for all $y \in \Psi(a)$ and $y < 0$ for all $y \in \Psi(b)$. Then let

$$B = \{x \in [a, b] : y \geq 0 \text{ for all } y \in \Psi(x)\}$$

Since $a \in B$, $B \neq \emptyset$. Since $B \subseteq [a, b]$, B is bounded. Then let

$$c = \sup B$$

Since B is nonempty and bounded, $c \in \mathbb{R}$, and $B \subseteq [a, b] \Rightarrow c \in [a, b]$. Now claim $0 \in \Psi(c)$.

To see this, first note that for every n there exists $x_n \in B$ such that

$$c - \frac{1}{n} \leq x_n \leq c$$

Then $x_n \rightarrow c$ by construction. Now let $y \in \Psi(c)$. Since Ψ is lhc at c, there exists $\{y_n\}$ such that $y_n \in \Psi(x_n)$ for each n and $y_n \rightarrow y$. Since $x_n \in B$ for each n, $y_n \geq 0$ for each n. Thus $y = \lim_n y_n \geq 0$. Since y was arbitrary, $y \geq 0$ for each $y \in \Psi(c)$. Then note that this implies $c \neq b$, so $c < b$.

Now suppose by way of contradiction that $0 \notin \Psi(c)$. Then $\Psi(c) \subseteq (0, \infty)$ and $V = (0, \infty)$ is an open set. Since Ψ is uhc at c, there exists an open set $W \ni c$ such that for all $x \in W$, $\Psi(x) \subseteq V = (0, \infty)$. But $c < b$ and $W \ni c$ is open, so there exists $x \in W \cap [a, b]$ with $x > c$. Since $x > c$, $x \notin B$, so there exists $y \in \Psi(x)$ such that $y < 0$. This is a contradiction. Therefore $0 \in \Psi(c)$.

b. Let $\Phi : [a, b] \rightarrow 2^{[a,b]}$ be a correspondence that is continuous, and nonempty- and closed-valued, so $\Phi(x) \subseteq [a, b]$ is a nonempty closed set for each $x \in [a, b]$. Show that Φ has a fixed point.

(**Hint:** No theorem from class will directly imply this result. Use (a).)

Solution: Let $\Psi : [a, b] \rightarrow 2^{\mathbb{R}}$ be given by

$$\Psi(x) = \Phi(x) - \{x\} \quad \text{for each } x \in [a, b]$$

Note that from (a) it suffices to show that Ψ is continuous and nonempty-valued, as in that case (a) shows $\exists x^* \in [a, b]$ such that $0 \in \Psi(x^*)$, which implies $0 \in \Phi(x^*) - \{x^*\}$ or $x^* \in \Phi(x^*)$.

Since $\Phi(x) \neq \emptyset$ for each x, $\Psi(x) \neq \emptyset$ for each x. To see that Ψ is continuous, first note that $\Phi(x) \subseteq [a, b]$ for each x, so $\Psi(x) \subseteq [a-b, b-a]$ for each x. Next note that $\Phi(x) \subseteq [a, b]$ is closed and hence compact for each x, so $\Psi(x)$ is also compact for each x. Then to show that Ψ is uhc, it suffices to show that Ψ has closed graph. To that end, let $(x_n, w_n) \in$ graph Ψ for each n and $(x_n, w_n) \to (x, w)$. Since $w_n \in \Psi(x_n)$ for each n, there exists $x_n \in \Phi(x_n)$ for each n such that $w_n = z_n - x_n$. Then $x_n \to x$ and $w_n \to w$, so $z_n = w_n + x_n \to w + x = z$. So $(x_n, z_n) \to (x, z)$ and $(x_n, z_n) \in$ graph Φ for each n. Since Φ is uhc and closed-valued, Φ has closed graph, so $(x, z) \in$ graph Φ, that is, $z \in \Phi(x)$. Thus $w = z - x \in$ graph Ψ. Thus Ψ has closed graph, and hence is uhc.

To see that Ψ is lhc, let $x \in [a, b]$ and $\{x_n\} \subseteq [a, b]$ with $x_n \to x$. Let $w \in \Psi(x)$, so $w = z - x$ for some $z \in \Phi(x)$. Since Φ is lhc at x, there exists $\{z_n\}$ with $z_n \in \Phi(x_n)$ for each n and $z_n \to z$. Then $z_n - x_n \to z - x = w$ and $z_n - x_n \in \Psi(x_n)$ for each n. Thus Ψ is lhc.

Here is an argument that Ψ is uhc using the sequential characterization of uhc. First note that Ψ is compact-valued, so the sequential characterization of uhc is valid. Then let $\{x_n\} \subseteq [a, b]$ with $x_n \to x$ and $w_n \in \Psi(x_n)$ for each n. For each n there exists $z_n \in \Phi(x_n)$ such that $w_n = z_n - x_n$. Since Φ is uhc and compact-valued, there exists a subsequence $\{z_{n_k}\}$ such that $z_{n_k} \to z \in \Phi(x)$. Then $w_{n_k} = z_{n_k} - x_{n_k} \to z - x \in \Psi(x)$. Therefore Ψ is uhc.

Here is an argument that Ψ is uhc using the definition. Let V be an open set such that $\Psi(x) \subseteq V$. Then since $\Psi(x)$ is compact there exists $\varepsilon > 0$ such that

$$B_\varepsilon(\Psi(x)) = B_\varepsilon(\Phi(x) - \{x\}) \subseteq V$$

Then $B_{\frac{\varepsilon}{2}}(\Phi(x))$ is an open set and $\Phi(x) \subseteq B_{\frac{\varepsilon}{2}}(\Phi(x))$, so using uhc of Φ there exists an open set $W \ni x$ such that

$$\Phi(y) \subseteq B_{\frac{\varepsilon}{2}}(\Phi(x)) \quad \forall y \in W$$

Then $x \in W$ and W is open, so there exists $\delta > 0$ such that $B_{\frac{\delta}{2}}(x) \subseteq W$. Then set $\bar{\varepsilon} = \min\{\frac{\delta}{2}, \frac{\varepsilon}{2}\}$. $B_{\bar{\varepsilon}}(x) \ni x$ is open and for all $y \in B_{\bar{\varepsilon}}(x) \subseteq W$, if $w \in \Psi(y)$ then $w = z - y$ for some $z \in \Phi(y) \subseteq B_{\frac{\varepsilon}{2}}(\Phi(x))$. Thus there exists $z' \in \Phi(x)$ such that $d(z, z') < \frac{\varepsilon}{2}$. So

$$\begin{aligned} d(z - y, z' - x) &\leq d(z, z') + d(y, x) \\ &< \frac{\varepsilon}{2} + \bar{\varepsilon} \\ &\leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \end{aligned}$$

So $w = z - y \in B_\varepsilon(\Psi(x)) \subseteq V$. Since $w \in \Psi(y)$ was arbitrary, $\Psi(y) \subseteq V$.

Finally, here is an argument that Ψ is lhc using the definition. Let V be an open set such that $\Psi(x) \cap V \neq \emptyset$. Then $(\Phi(x) - \{x\}) \cap V \neq \emptyset$, so there exists $z \in \Phi(x)$ such that $z - x \in V$. Since V is open, there exists $\varepsilon > 0$ such that $B_\varepsilon(z - x) \subseteq V$. Then $B_{\frac{\varepsilon}{2}}(z)$ is open and $\Phi(x) \cap B_{\frac{\varepsilon}{2}}(z) \neq \emptyset$, so using the lhc of

Φ, there exists an open set $W \ni x$ such that for all $y \in W$, $\Phi(y) \cap B_{\frac{\varepsilon}{2}}(z) \neq \emptyset$. Since W is open and $x \in W$, there exists $\delta > 0$ such that $B_{\frac{\delta}{2}}(x) \subseteq W$. Then let $\bar{\varepsilon} = \min(\frac{\delta}{2}, \frac{\varepsilon}{2})$. $B_{\bar{\varepsilon}}(x) \ni x$ is open, and for all $y \in B_{\bar{\varepsilon}}(x)$ there exists $z' \in \Phi(y)$ such that $d(z', z) < \frac{\varepsilon}{2}$. Then

$$\begin{aligned} d(z' - y, z - x) &\leq d(z', z) + d(y, x) \\ &< \frac{\varepsilon}{2} + \bar{\varepsilon} \\ &\leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \end{aligned}$$

So $z' - y \in \Psi(y) \cap B_{\varepsilon}(z - x)$ and $B_{\varepsilon}(z - x) \subseteq V$, so $\Psi(y) \cap V \neq \emptyset$.