Economics 204 Summer/Fall 2019
Final Exam

Answer all of the questions below. Be as complete, correct, and concise as possible. There are 6 questions for a total of 165 points possible; point values for each problem are in parentheses. For questions with subparts, each subpart is worth the same number of points. You have 180 minutes to complete the exam. Use the points as a guide to allocating your time. You may use any result from class with appropriate references unless you are specifically being asked to prove it.

Do not turn the page until the exam begins.

1. (15) Define or state each of the following.

   (a) eigenvalue of a linear transformation $T : X \to Y$ between vector spaces X and Y over the same field F

   (b) Cauchy sequence in a metric space (X, d)

   (c) Separating Hyperplane Theorem

2. (30) Show that for every $n \in \mathbb{N} = \{1, 2, 3, \ldots\}$,

$$\sum_{k=1}^{n}(2k-1) = n^2$$

3. (30) Let X and Y be vector spaces over the same field F, and let $T : X \to Y$ be a linear transformation.

   (a.) Show that $\ker T$ is a vector subspace of X, and $\operatorname{Im} T$ is a vector subspace of Y.

   (b.) Suppose X is finite-dimensional. Show that $\dim X = \dim \ker T + \operatorname{Rank} T$.

4. (30) Let $X \subseteq \mathbb{R}^n$ be open and $f : X \to \mathbb{R}$ be differentiable on X. Suppose $x^* \in X$ and $f(x^*) \geq f(x)$ for all $x \in X$. Show that $Df(x^*) = 0$.

5. (30) Let (X, d) be a metric space and $C \subseteq X$ be compact. Let $U \subseteq X$ be an open set such that $C \subseteq U$. Show that there exists $\varepsilon > 0$ such that

$$B_\varepsilon(C) = \bigcup_{x \in C} B_\varepsilon(x) \subseteq U$$

6. (30) Let $a, b \in \mathbb{R}$ with $a \leq b$.

   a. Let $\Psi : [a, b] \to 2^{\mathbb{R}}$ be a correspondence that is continuous and nonempty-valued, so $\Psi(x) \subseteq \mathbb{R}$ is a nonempty set for each $x \in [a, b]$. Suppose $y \geq 0 \ \forall\, y \in \Psi(a)$, and $y \leq 0 \ \forall\, y \in \Psi(b)$ (so $\Psi(a) \subseteq [0, \infty)$ and $\Psi(b) \subseteq (-\infty, 0]$). Show that there exists $c \in [a, b]$ such that $0 \in \Psi(c)$.

   b. Let $\Phi : [a, b] \to 2^{[a,b]}$ be a correspondence that is continuous, and nonempty- and closed-valued, so $\Phi(x) \subseteq [a, b]$ is a nonempty closed set for each $x \in [a, b]$. Show that Φ has a fixed point.

      (**Hint:** No theorem from class will directly imply this result. Use (a).)