Econ 204 2020

Lecture 12

Outline

1. An Example
2. Regular and Critical Points and Values
3. Inverse Function Theorem
4. Implicit Function Theorem
5. Sard's Theorem
6. Transversality Theorem

Comparative Statics

In many problems we are interested in how endogenously determined variables are affected by exogenously given parameters. Here we study problems in which the variables of interest are characterized as solutions to a parameterized family of equations.

To formalize, let $X \subseteq \mathbf{R}^n$ and $A \subseteq \mathbf{R}^p$ be open, and let $f : X \times A \to \mathbf{R}^m$. For a given $a \in A$, consider solutions $x \in X$ to the family of equations

$$f(x, a) = 0$$

We want to characterize the set of solutions and study how this set depends on the parameter a.

An Example

Consider the function $f : (0, 2\pi) \times \mathbf{R} \to \mathbf{R}$ defined by

$$f(x, a) = \sin x + a$$

Let $X = (0, 2\pi)$. For fixed a, let

$$f_a(x) = f(x, a) = \sin x + a$$

We look for solutions $x \in (0, 2\pi)$ to the equation

$$f_a(x) = f(x, a) = \sin x + a = 0$$

that is, the $x \in (0, 2\pi)$ such that

$$\sin x = -a$$

Let $\Psi : A \to 2^X$ denote the solution correspondence, so

$$\Psi(a) = \{x \in (0, 2\pi) : f_a(x) = \sin x + a = 0\}$$

An Example

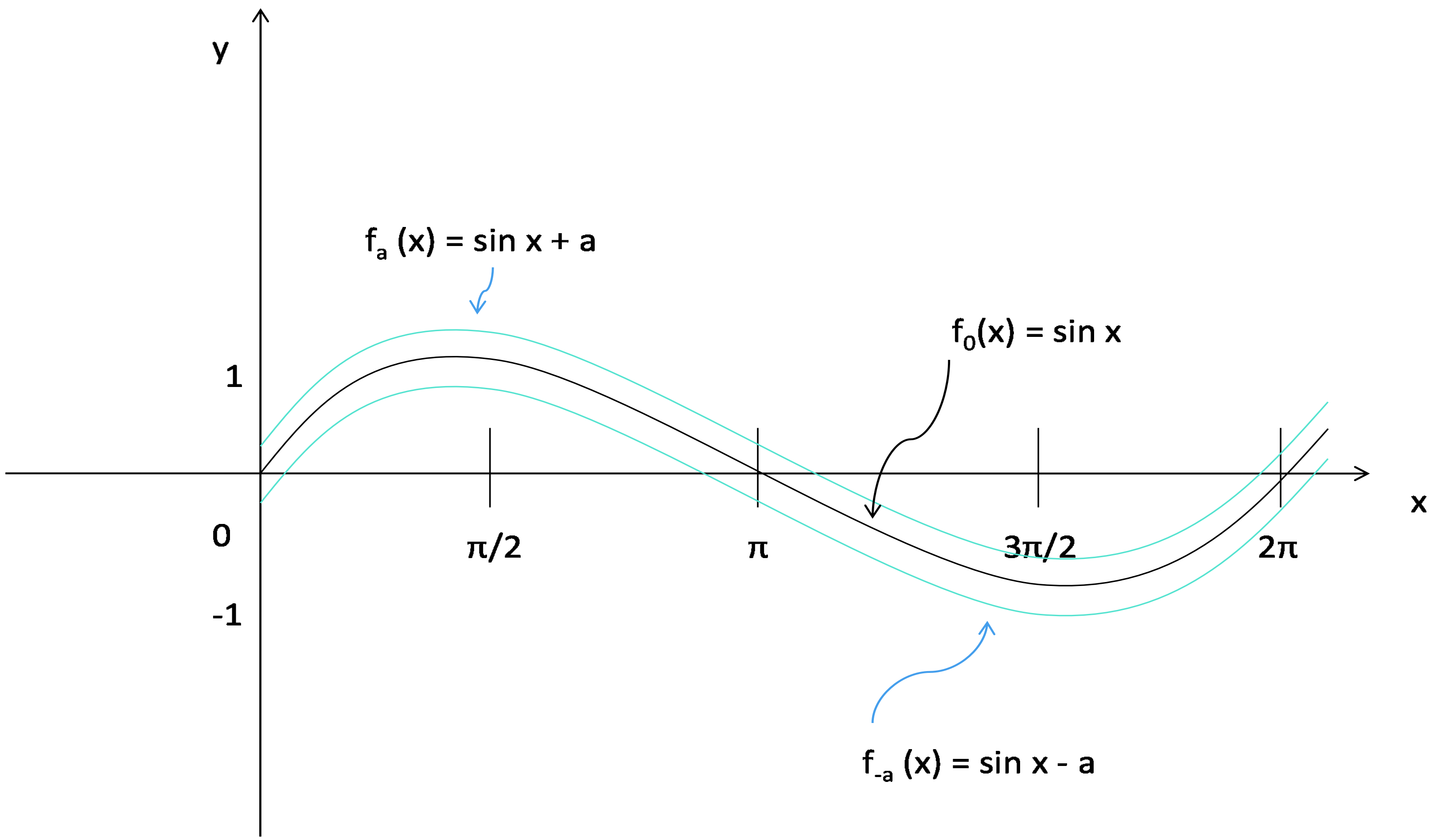
Start with $a = 0$. For $x \in (0, 2\pi)$,

$$f_0(x) = \sin x = 0 \iff x = \pi$$

so $\Psi(0) = \{\pi\}$.

Notice that for x near π, for example in the neighborhood $(\pi/2, 3\pi/2)$, and for a near 0, $\sin^{-1}(a)$ remains single-valued and depends smoothly on a.

In addition, we can predict the direction of change: x is increasing in a.


y
$f_a(x) = \sin x + a$
$f_0(x) = \sin x$
1
0
-1
π/2
π
3π/2
2π
x
$f_{-a}(x) = \sin x - a$

An Example

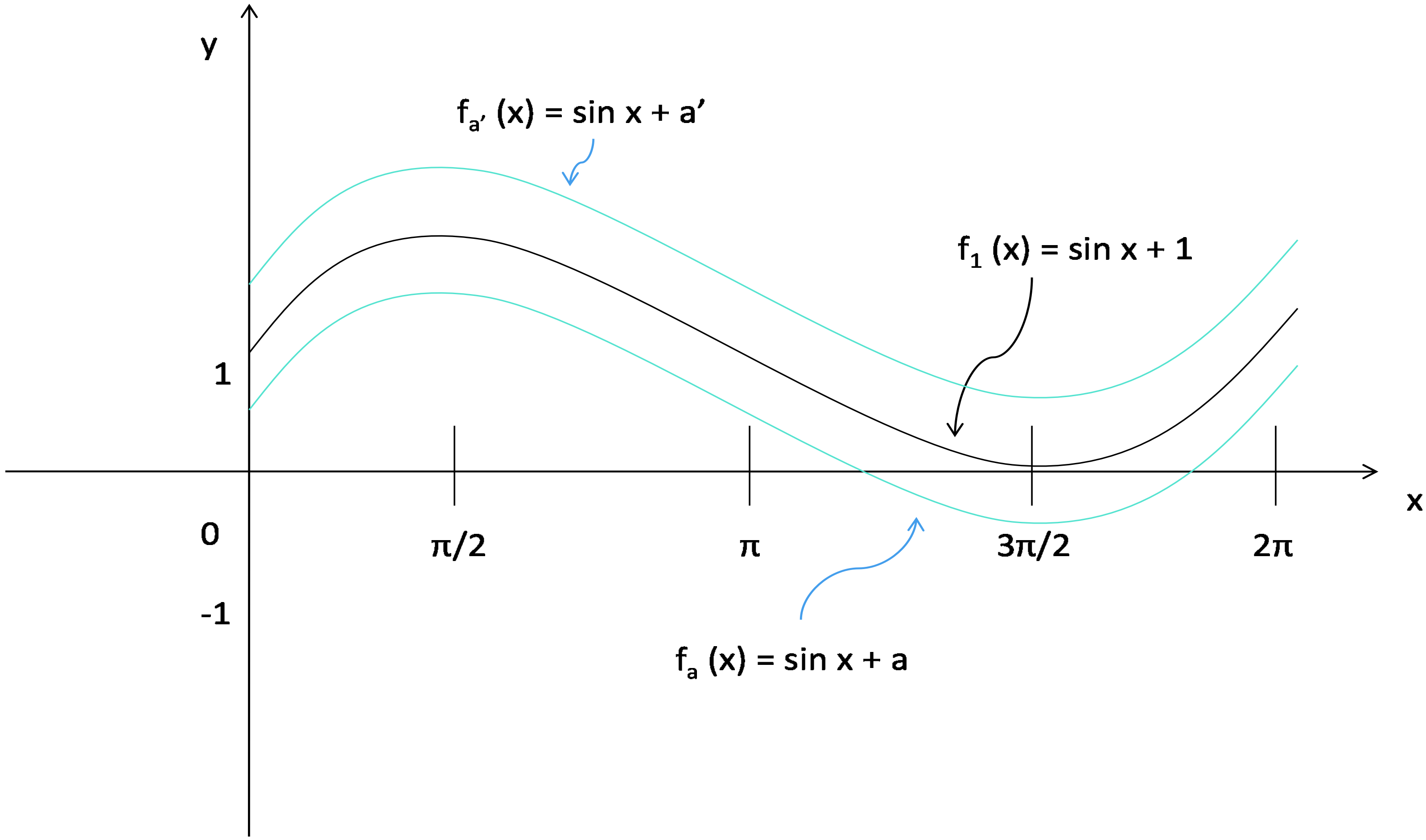
Now consider $a = 1$. For $x \in (0, 2\pi)$,

$$\begin{aligned} f_1(x) &= \sin x + 1 = 0 \\ &\iff \sin x = -1 \\ &\iff x = \frac{3\pi}{2} \end{aligned}$$

So $\Psi(1) = \{3\pi/2\}$.

But note that for $a' > 1$, $\Psi(a') = \emptyset$, while for $a < 1$ close to 1, there are two solutions near $3\pi/2$, one above and one below $3\pi/2$.

Ψ is not lower hemicontinuous at $a = 1$.


y
$f_{a'}(x) = \sin x + a'$
$f_1(x) = \sin x + 1$
1
0
-1
$\pi/2$
π
$3\pi/2$
2π
x
$f_a(x) = \sin x + a$

Regular and Critical Points and Values

Suppose $X \subseteq \mathbf{R}^n$ is open. Suppose $f : X \to \mathbf{R}^m$ is differentiable at $x \in X$, and let $W = \{e_1, \ldots, e_n\}$ denote the standard basis of $\mathbf{R}^n$. Then $df_x \in L(\mathbf{R}^n, \mathbf{R}^m)$, and

$$\begin{aligned}
\text{Rank}\, df_x &= \dim \text{Im}\,(df_x) \\
&= \dim \text{span}\,\{df_x(e_1), \ldots, df_x(e_n)\} \\
&= \dim \text{span}\,\{Df(x)e_1, \ldots, Df(x)e_n\} \\
&= \dim \text{span}\,\{\text{column 1 of } Df(x), \ldots, \text{column } n \text{ of } Df(x)\} \\
&= \text{Rank}\, Df(x)
\end{aligned}$$

Thus,

$$\text{Rank}\, df_x \leq \min\{m, n\}$$

We say df_x has *full rank* if $\text{Rank}\, df_x = \min\{m, n\}$, that is, is df_x has maximum possible rank.

Regular and Critical Points and Values

Definition 1. *Suppose $X \subseteq \mathbf{R}^n$ is open. Suppose $f : X \to \mathbf{R}^m$ is differentiable on X.*

- x *is a* regular point *of* f *if* Rank $df_x = \min\{m, n\}$.

- x *is a* critical point *of* f *if* Rank $df_x < \min\{m, n\}$.

- y *is a* critical value *of* f *if there exists* $x \in f^{-1}(y)$ *such that* x *is a critical point of* f.

- y *is a* regular value *of* f *if* y *is not a critical value of* f

Example: Consider the function $g : (0, 2\pi) \to \mathbf{R}$ defined by

$$g(x)(= f_0(x)) = \sin x$$

Note that $g'(x) = \cos x$, so $g'(x) = 0 \iff x = \pi/2$ or $x = 3\pi/2$. $Dg(x)$ is the 1×1 matrix $(g'(x))$, so

$$\operatorname{Rank} dg_x = \operatorname{Rank} Dg(x) = 1 \iff g'(x) \neq 0$$

critical points of g: $\pi/2$ and $3\pi/2$

regular points of g: $\left(0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \frac{3\pi}{2}\right) \cup \left(\frac{3\pi}{2}, 2\pi\right)$

critical values of g: $g(\pi/2) = \sin(\pi/2) = 1$ and $g(3\pi/2) = \sin(3\pi/2) = -1$

regular values of g: $(-\infty, -1) \cup (-1, 1) \cup (1, \infty)$

In particular, notice that 0 is not a critical value of g.

Given $a \in \mathbf{R}$, as above consider the perturbed function

$$f_a(x) = g(x) + a$$

Notice that $f_a'(x) = g'(x)$, so the critical points of f_a are the same as those of g, $\pi/2$ and $3\pi/2$.

For a close to zero, the solution to the equation

$$f_a(x) = 0$$

near $x = \pi$ moves smoothly with respect to changes in a. The direction the solution moves is determined by the sign of f_a'.

Now let $a = 1$. Since $3\pi/2$ is a critical point of f_1, 0 is a critical value of f_1.

Inverse Function Theorem

Theorem 1 (Thm. 4.6, Inverse Function Theorem)**.** *Suppose $X \subseteq \mathbf{R}^n$ is open, $f : X \to \mathbf{R}^n$ is C^1 on X, and $x_0 \in X$. If* $\det Df(x_0) \neq 0$ *(i.e. x_0 is a regular point of f) then there are open neighborhoods U of x_0 and V of $f(x_0)$ such that*

$$\begin{aligned} f : U \to V & \quad \text{is} \quad \text{one-to-one and onto} \\ f^{-1} : V \to U & \quad \text{is} \quad C^1 \\ Df^{-1}(f(x_0)) & \quad = \quad [Df(x_0)]^{-1} \end{aligned}$$

If in addition $f \in C^k$, then $f^{-1} \in C^k$.

Remark: f is one-to-one only on U; it need not be one-to-one globally. Thus f^{-1} is only a local inverse.

Proof. Read the proof in de la Fuente. This is pretty hard. The idea is that since $\det Df(x_0) \neq 0$, then $df_{x_0} : \mathbf{R}^n \to \mathbf{R}^n$ is one-to-one and onto. You need to find a neighborhood U of x_0 sufficiently small such that the Contraction Mapping Theorem implies that f is one-to-one and onto.

To see the formula for Df^{-1}, let id_U denote the identity function from U to U and I denote the $n \times n$ identity matrix. Then

$$\begin{aligned} Df^{-1}(f(x_0))Df(x_0) &= D(f^{-1} \circ f)(x_0) \\ &= D(\mathrm{id}_U(x_0)) \\ &= I \\ \Rightarrow Df^{-1}(f(x_0)) &= [Df(x_0)]^{-1} \end{aligned}$$

$\square$

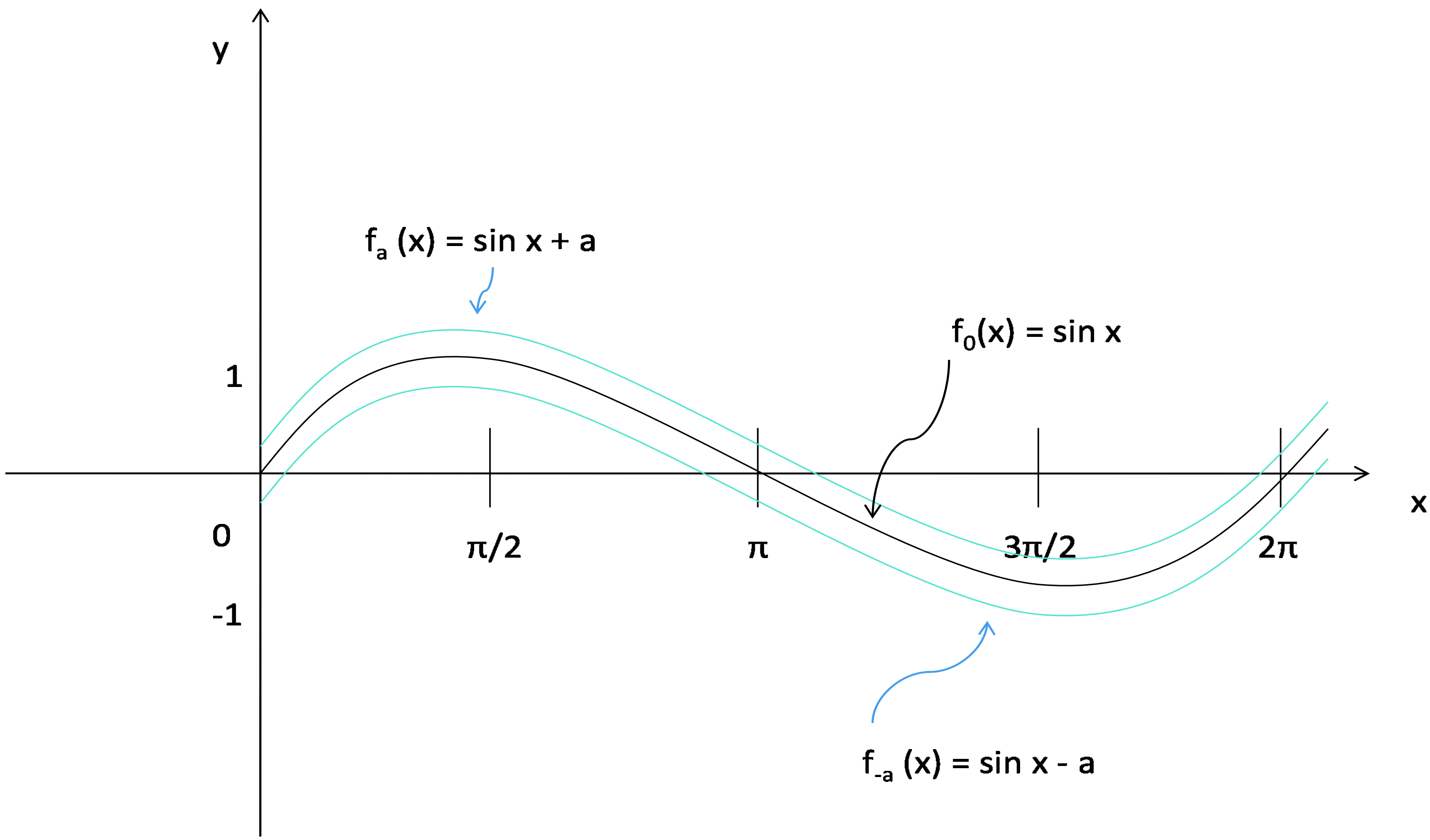
Inverse Function Theorem

Example: Let $g : (0, 2\pi) \to \mathbf{R}$ be given by $g(x) = \sin x$. Let $x_0 = \pi$.

Then $g'(x_0) = \cos\pi = -1 \neq 0$, so by the Inverse Function Theorem there exists an open set $U \subseteq (0, 2\pi)$ with $\pi \in U$, an open set $V \subseteq \mathbf{R}$ with $0 = g(\pi) \in V$ and a C^1 function $h : V \to U$ such that $g(h(v)) = v$ for all $v \in V$.

At $x = 3\pi/2$, $g'(x) = \cos(3\pi/2) = 0$, and g has no local inverse function there: for every open neighborhood U of $3\pi/2$ and every open neighborhood V of $-1 = g(3\pi/2)$, there exists $v \in V$ and $x_1 \neq x_2 \in U$ such that

$$g(x_1) = \sin x_1 = v = \sin x_2 = g(x_2)$$


y
$f_a(x) = \sin x + a$
$f_0(x) = \sin x$
1
0
-1
$\pi/2$
π
$3\pi/2$
2π
x
$f_{-a}(x) = \sin x - a$

Implicit Function Theorem

Theorem 2 (Thm. 2.2, Implicit Function Theorem)**.** *Suppose $X \subseteq \mathbf{R}^n$ and $A \subseteq \mathbf{R}^p$ are open and $f : X \times A \to \mathbf{R}^n$ is C^1. Suppose $f(x_0, a_0) = 0$ and $\det(D_x f(x_0, a_0)) \neq 0$, i.e. x_0 is a regular point of $f(\cdot, a_0)$. Then there are open neighborhoods U of x_0 ($U \subseteq X$) and W of a_0 such that*

$$\forall a \in W \quad \exists! x \in U \text{ s.t. } f(x, a) = 0$$

For each $a \in W$ let $g(a)$ be that unique x. Then $g : W \to X$ is C^1 and

$$Dg(a_0) = -\left[D_x f(x_0, a_0)\right]^{-1} \left[D_a f(x_0, a_0)\right]$$

If in addition $f \in C^k$, then $g \in C^k$.

Proof. Use the Inverse Function Theorem in the right way. Why is the formula for Dg correct? Assuming the implicit function exists and is differentiable,

$$\begin{aligned} 0 &= Df(g(a), a)(a_0) \\ &= D_x f(x_0, a_0) Dg(a_0) + D_a f(x_0, a_0) \\ Dg(a_0) &= -[D_x f(x_0, a_0)]^{-1} D_a f(x_0, a_0) \end{aligned}$$

$\square$

Implicit Function Theorem

Corollary 1. *Suppose* $X \subseteq \mathbf{R}^n$ *and* $A \subseteq \mathbf{R}^p$ *are open and* $f : X \times A \to \mathbf{R}^n$ *is* C^1. *If 0 is a regular value of* $f(\cdot, a_0)$, *then the correspondence*

$$a \mapsto \{x \in X : f(x, a) = 0\}$$

is lower hemicontinuous at a_0.

Proof. If 0 is a regular value of $f(\cdot, a_0)$, then given any $x_0 \in \{x \in X : f(x, a_0) = 0\}$, we can find a local implicit function g; in other words, if a is sufficiently close to a_0, then $g(a) \in \{x \in X : f(x, a) = 0\}$; the continuity of g then shows that the correspondence $\{x \in X : f(x, a) = 0\}$ is lower hemicontinuous at a_0. $\square$

Implicit Function Theorem

Example: Back to our opening example: $f : (0, 2\pi) \times \mathbf{R} \to \mathbf{R}$ given by $f(x, a) = \sin x + a$. Let $x_0 = \pi$ and $a_0 = 0$.

Then $f(x_0, a_0) = \sin \pi = 0$ and $D_x f(x_0, a_0) = \cos \pi = -1 \neq 0$. So $x_0 = \pi$ is a regular point of $f(\cdot, a_0)$.

By the Implicit Function Theorem, $\exists$ open neighborhoods $U \ni \pi$ and $W \ni 0$ and a C^1 function $h : W \to U$ such that $f(h(a), a) = 0$ for every $a \in W$, and

$$Dh(a_0) = -[\cos \pi]^{-1} \cdot 1 = 1$$

So the local solution is increasing in a near a_0 (as we saw above).

Again at $x = 3\pi/2$ and $a = 1$, $D_x f(x, a) = 0$ and no local implicit function exists:

for every open neighborhood U of $3\pi/2$ and W of 1, for any $a' > 1$ there is no $x' \in U$ such that $f(x', a') = \sin x' + a' = 0$.

Lebesgue Measure Zero

Definition 2. *Suppose* $A \subseteq \mathbf{R}^n$*.* A *has* Lebesgue measure zero *if for every* $\varepsilon > 0$ *there is a countable collection of rectangles* $I_1, I_2, \ldots$ *such that*

$$\sum_{k=1}^{\infty} \textit{Vol } (I_k) < \varepsilon \textit{ and } A \subseteq \bigcup_{k=1}^{\infty} I_k$$

Here by a rectangle we mean $I_k = \times_{j=1}^{n}(a_j^k, b_j^k)$ for some $a_j^k < b_j^k \in \mathbf{R}$, and

$$\text{Vol } (I_k) = \prod_{j=1}^{n} |b_j^k - a_j^k|$$

Notice that this defines Lebesgue measure zero without defining Lebesgue measure.

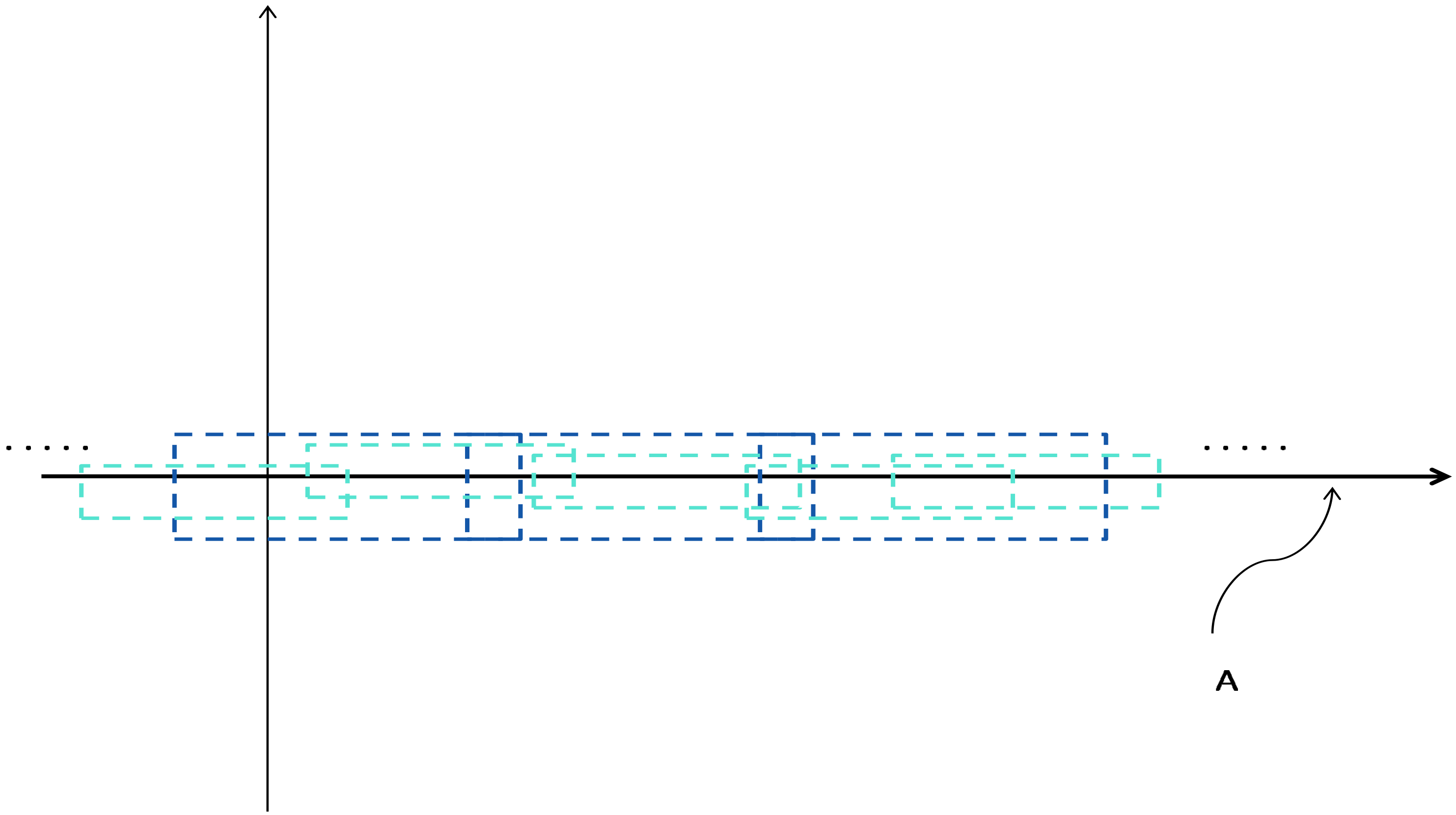
Lebesgue Measure Zero

Examples:

1. "Lower-dimensional" sets have Lebesgue measure zero. For example,

$$A = \{x \in \mathbf{R}^2 : x_2 = 0\}$$

has measure zero.


A

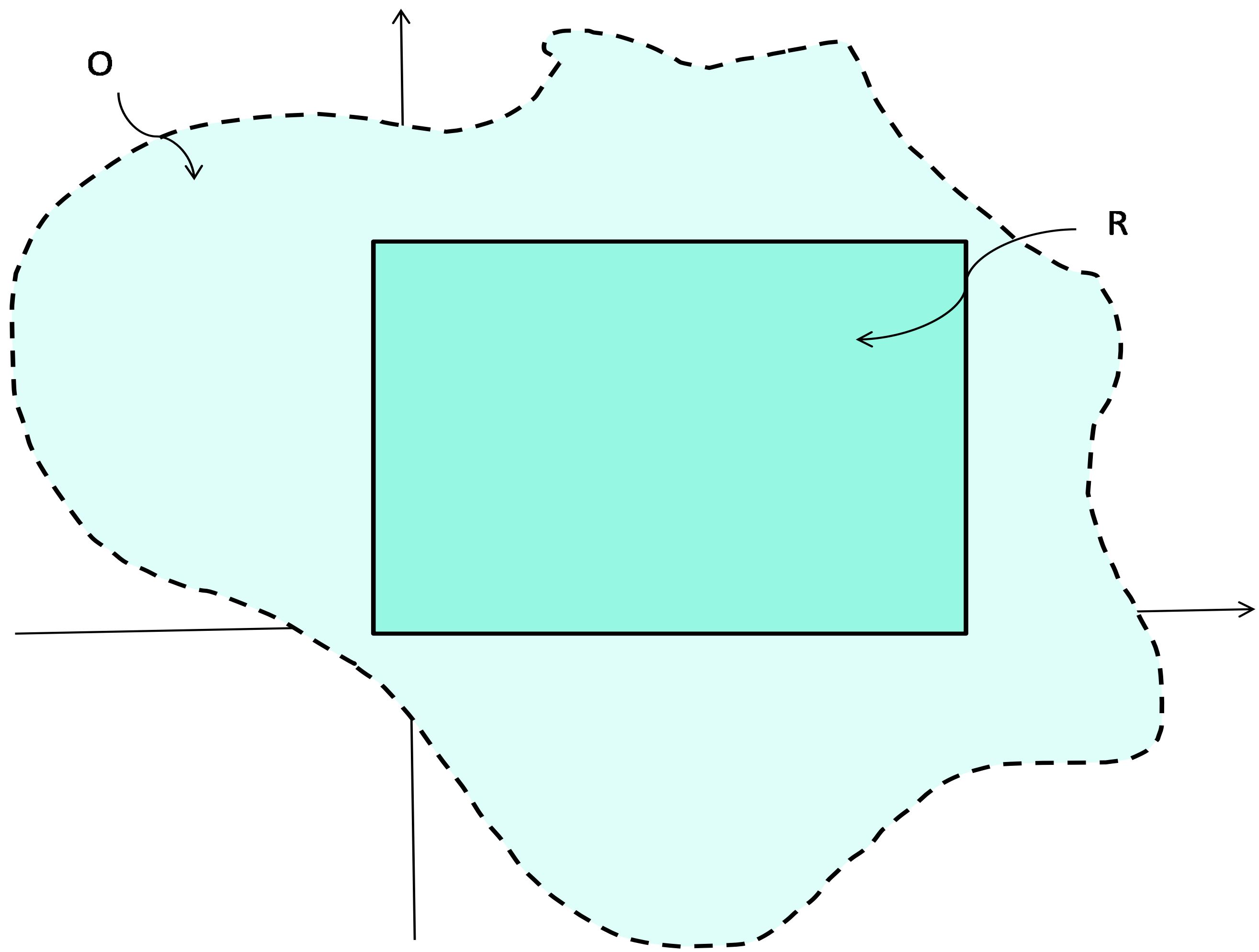
2. Any finite set has Lebesgue measure zero in $\mathbf{R}^n$.

3. If A_n has Lebesgue measure zero $\forall n$ then $\cup_{n \in \mathbf{N}} A_n$ has Lebesgue measure zero.

4. $\mathbf{Q}$ and every countable set has Lebesgue measure zero.

5. No open set in $\mathbf{R}^n$ has Lebesgue measure zero.

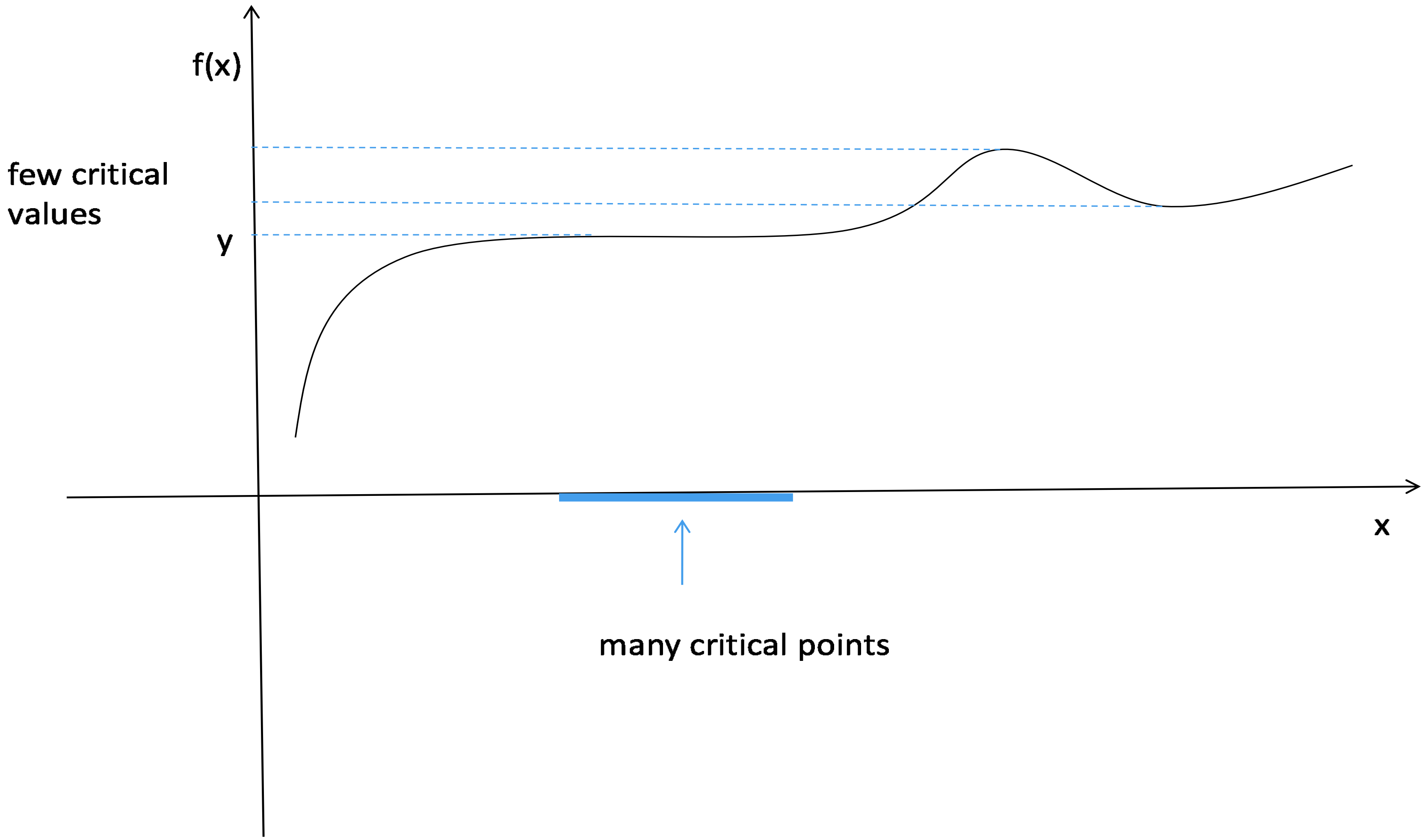
If $O \subset \mathbf{R}^n$ is open, then there exists a rectangle R such that $\bar{R} \subseteq O$ and such that Vol $(R) = r > 0$. If $\{I_j\}$ is any collection of rectangles such that $O \subseteq \cup_{j=1}^{\infty} I_j$, then $\bar{R} \subseteq O \subseteq \cup_{j=1}^{\infty} I_j$, so $\sum_{j=1}^{\infty}$ Vol $(I_j) \geq$ Vol $(R) = r > 0$.


O
R

Genericity

Lebesgue measure zero is a natural formulation of the notion that A is a small set. Without specifying a probability measure explicitly, this expresses the idea that if $x \in \mathbf{R}^n$ is chosen at random, then the probability that $x \in A$ is zero.

A function may have many critical points; for example, if a function is constant on an interval, then every element of the interval is a critical point. But it can't have many critical *values*.


f(x)
few critical
values
y
x
many critical points

Sard's Theorem

Theorem 3 (Thm. 2.4, Sard's Theorem)**.** *Let $X \subseteq \mathbf{R}^n$ be open, and $f : X \to \mathbf{R}^m$ be C^r with $r \geq 1 + \max\{0, n - m\}$. Then the set of all critical values of f has Lebesgue measure zero.*

Proof. First, we give a false proof that conveys the essential idea as to why the theorem is true; it can be turned into a correct proof. Suppose $m = n$. Let C be the set of critical points of f, V the set of critical values. Then

$$\begin{aligned}
\text{Vol}\,(V) &= \text{Vol}\,(f(C)) \\
&\leq \int_C |\det Df(x)|\,dx \text{ (equality if } f \text{ is one-to-one)} \\
&= \int_C 0\,dx \\
&= 0
\end{aligned}$$

Now, we outline how to turn this into a proof. First, show that we can write $X = \cup_{j \in \mathbf{N}} X_j$, where each X_j is a compact subset of $[-j, j]^n$. Let $C_j = C \cap X_j$. Fix j for now. Since f is C^1,

$$\begin{aligned} x_k \to x \quad &\Rightarrow \quad \det Df(x_k) \to \det Df(x) \\ \{x_k\} \subseteq C_j, x_k \to x \quad &\Rightarrow \quad \det Df(x) = 0 \Rightarrow x \in C_j \end{aligned}$$

so C_j is closed, hence compact. Since X is open and C_j is compact, there exists $\delta_1 > 0$ such that

$$B_{\delta_1}[C_j] = \cup_{x \in C_j} B_{\delta_1}[x] \subseteq X$$

$B_{\delta_1}[C_j]$ is bounded, and, using the compactness of C_j, one can show it is closed, so it is compact. Since $\det Df(x)$ is continuous on $B_{\delta_1}[C_j]$, it is uniformly continuous on $B_{\delta_1}[C_j]$. Then given $\varepsilon > 0$, we can find $\delta \leq \delta_1$ such that $B_\delta[C_j] \subseteq [-2j, 2j]^n$ and

$$x \in B_\delta[C_j] \Rightarrow |\det Df(x)| \leq \frac{\varepsilon}{2 \cdot 4^n j^n}$$

Then

$$
\begin{aligned}
f(C_j) &\subseteq f(B_\delta[C_j]) \\
\text{Vol}\ (f(B_\delta[C_j])) &\leq \int_{[-2j,2j]^n} \frac{\varepsilon}{2 \cdot 4^n j^n}\, dx \\
&= \frac{\varepsilon}{2}
\end{aligned}
$$

Since f is C^1, show that $f(C_j)$ can be covered by a countable collection of rectangles of total volume less than ε. Since $\varepsilon > 0$ is arbitrary, $f(C_j)$ has Lebesgue measure zero. Then

$$f(C) = f\left(\cup_{j\in\mathbf{N}} C_j\right) = \cup_{n\in\mathbf{N}} f(C_j)$$

is a countable union of sets of Lebesgue measure zero, so $f(C)$ has Lebesgue measure zero. $\square$

Sard's Theorem

Remark: Sard's Theorem has a number of powerful implications. Given a randomly chosen function f, it is very unlikely that zero will be a critical value of f. If by some fluke zero *is* a critical value of f, then a slight perturbation of f will make zero a regular value. We return to a more wide-ranging version of this statement below.

Example: Let $g : (0, 2\pi) \to \mathbf{R}$ be given by $g(x) = \sin x$. We calculated by hand above that the set of critical values of g is $\{-1, 1\}$. Since this set is finite, it has Lebesgue measure zero.

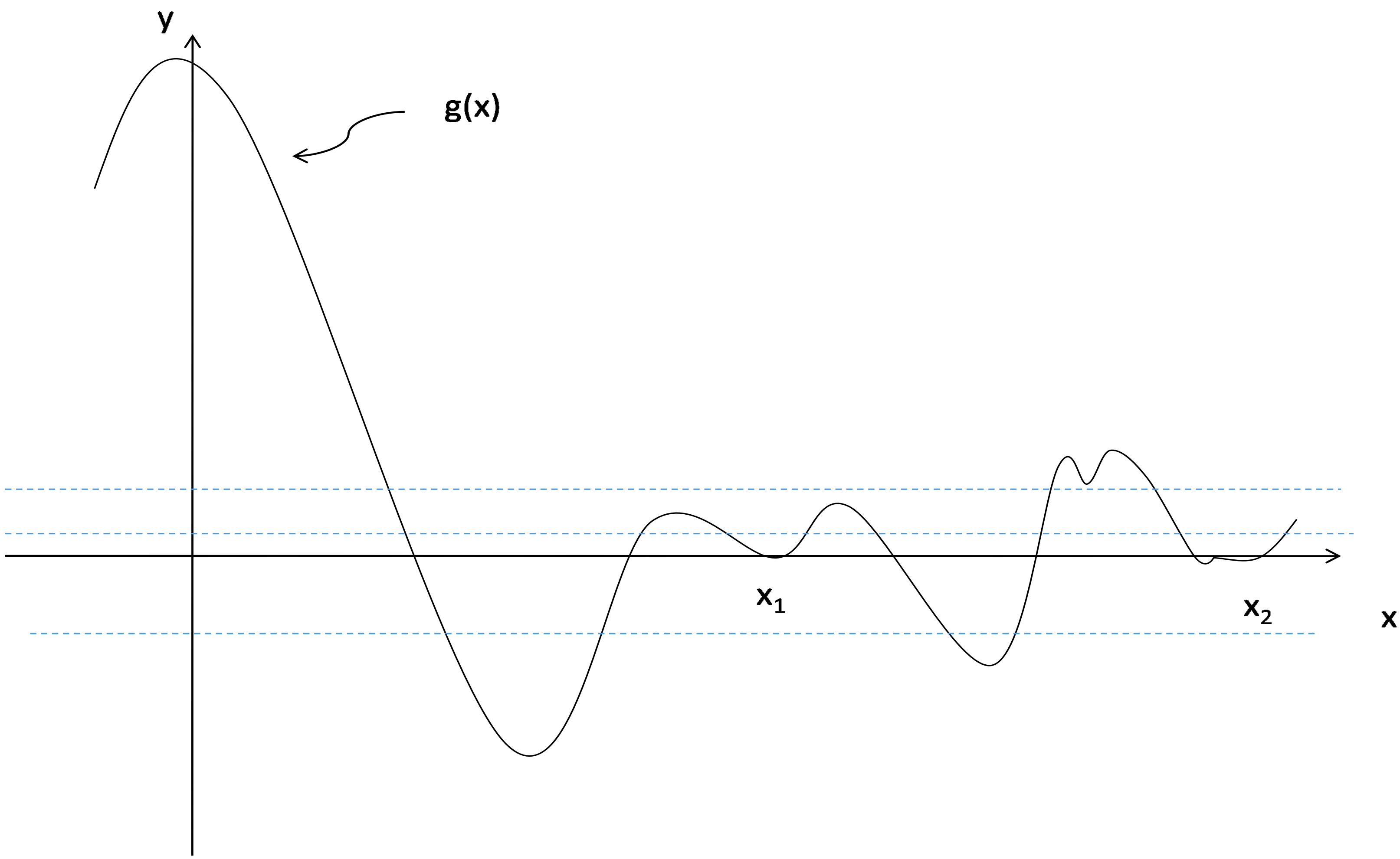
Transversality

Let $g : \mathbf{R}^n \to \mathbf{R}^n$ be C^1. Consider the family of n equations in n variables:

$$g(x) = 0$$

Suppose for some x such that $g(x) = 0$, rank $(Dg(x)) < n$. That is, some $x \in g^{-1}(0)$ is a critical point of g, thus 0 is a critical value of g.

By Sard's Theorem, almost every $a \neq 0$ is a regular value of g. So for a outside a set of Lebesgue measure 0, $Dg(x)$ has full rank for every x solving $g(x) = a$. For any such a and any $x \in g^{-1}(a)$, we can use the Inverse Function Theorem to show that a local inverse $x(a)$ exists, and give a formula for $Dx(a)$.


y
g(x)
x1
x2
x

Transversality

Suppose $f : \mathbf{R}^n \times \mathbf{R}^p \to \mathbf{R}^m$. We care about the parameterized family of equations

$$f(x, a) = 0$$

where, as above, we interpret $a \in \mathbf{R}^p$ to be a vector of parameters that indexes the function $f(\cdot, a)$.

For a given a, we are interested in the set of solutions

$$\{x \in X : f(x, a) = 0\}$$

and the way that this correspondence depends on a.

If f is separable in a, that is, $f(x, a) = g(x) + a$, then we can use Sard's Theorem (PS6 2010).

Transversality Theorem

Separability is strong, and not required: If f depends on a in a nonseparable fashion, it is enough that from any solution $f(x, a) = 0$, any directional change in f can be achieved by arbitrarily small changes in x and a.

Theorem 4 (Thm. 2.5', Transversality Theorem)**.** *Let $X \subseteq \mathbf{R}^n$ and $A \subseteq \mathbf{R}^p$ be open, and $f : X \times A \to \mathbf{R}^m$ be C^r with $r \geq 1 + \max\{0, n - m\}$. Suppose that 0 is a regular value of f. Then there is a set $A_0 \subseteq A$ such that $A \setminus A_0$ has Lebesgue measure zero and for all $a \in A_0$, 0 is a regular value of $f_a = f(\cdot, a)$.*

Remark: Notice the important difference between the statement that 0 is a regular value of f (one of the assumptions of the Transversality Theorem), and the statement that 0 is a regular value of f_a for a fixed $a \in A_0$ (part of the conclusion of the Transversality Theorem). 0 is a regular value of f if and only if $Df(x,a)$ has full rank for every (x,a) such that $f(x,a) = 0$. Instead, for fixed $a_0 \in A_0$, 0 is a regular value of $f_{a_0} = f(\cdot, a_0)$ if and only if $D_x f(x, a_0)$ has full rank for every x such that $f_{a_0}(x) = f(x, a_0) = 0$.

Remark: Consider the important special case in which $n = m$, so we have as many equations (m) as endogenous variables (n). In this case, suppose f is C^1 (note that $1 = 1 + \max\{0, n - n\}$). If 0 is a regular value of f, that is, $Df(x, a)$ has rank $n = m$ for every (x, a) such that $f(x, a) = 0$, then by the Transversality Theorem there is a set $A_0 \subset A$ such that $A \setminus A_0$ has Lebesgue measure zero and for every $a_0 \in A_0$, $D_x f(x, a_0)$ has rank $n = m$ for all x such that $f(x, a_0) = 0$.

Fix $a_0 \in A_0$ and x_0 such that $f(x_0, a_0) = 0$. By the Implicit Function Theorem, there exist open sets A^* containing a_0 and X^* containing x_0, and a C^1 function $x : A^* \to X^*$ such that

- $x(a_0) = x_0$

- $f(x(a), a) = 0$ for every $a \in A^*$

- if $(x, a) \in X^* \times A^*$ then

$$f(x, a) = 0 \iff x = x(a)$$

  that is, x_0 is locally unique, and $x(a)$ is locally unique for each $a \in A^*$

- $Dx(a_0) = -[D_x f(x_0, a_0)]^{-1} D_a f(x_0, a_0)$

Transversality

Example: Back to the opening example: $f : (0, 2\pi) \times \mathbf{R} \to \mathbf{R}$ given by $f(x, a) = \sin x + a$.

For any (x, a) such that $f(x, a) = 0$, $Df(x, a) = (\cos x, 1)$ which has rank $1 = \min\{2, 1\}$. Thus 0 is a regular value of f.

Set $A_0 = \mathbf{R} \backslash \{-1, 1\}$. Since $\{-1, 1\}$ is a finite set, it has Lebesgue measure zero in $\mathbf{R}$.

Again we have already calculated by hand that for any $a \in A_0$, 0 is a regular value of $f_a = f(\cdot, a)$.