

Econ 204 2020

Lecture 13

Outline

0. *Transversality Theorem*

1. Fixed Points for Functions
2. Brouwer's Fixed Point Theorem
3. Fixed Points for Correspondences
4. Kakutani's Fixed Point Theorem
5. Separating Hyperplane Theorems

Transversality

Suppose $f : \mathbf{R}^n \times \mathbf{R}^p \rightarrow \mathbf{R}^m$. We care about the parameterized family of equations

$$f(x, a) = 0$$

where, as above, we interpret $a \in \mathbf{R}^p$ to be a vector of parameters that indexes the function $f(\cdot, a)$.

For a given a , we are interested in the set of solutions

$$\psi(a) = \{x \in X : f(x, a) = 0\}$$

and the way that this correspondence depends on a .

If f is separable in a , that is, $f(x, a) = g(x) + a$, then we can use Sard's Theorem (PS6 2010).

Transversality Theorem

Separability is strong, and not required: If f depends on a in a nonseparable fashion, it is enough that from any solution $f(x, a) = 0$, any directional change in f can be achieved by arbitrarily small changes in x and a .

Theorem 4 (Thm. 2.5', Transversality Theorem). *Let $X \subseteq \mathbf{R}^n$ and $A \subseteq \mathbf{R}^p$ be open, and $f : X \times A \rightarrow \mathbf{R}^m$ be C^r with $r \geq 1 + \max\{0, n - m\}$. Suppose that 0 is a regular value of f . Then there is a set $A_0 \subseteq A$ such that $A \setminus A_0$ has Lebesgue measure zero and for all $a \in A_0$, 0 is a regular value of $f_a = f(\cdot, a)$.*

Remark: Notice the important difference between the statement that 0 is a regular value of f (one of the assumptions of the Transversality Theorem), and the statement that 0 is a regular value of f_a for a fixed $a \in A_0$ (part of the conclusion of the Transversality Theorem). 0 is a regular value of f if and only if $Df(x, a)$ has full rank for every (x, a) such that $f(x, a) = 0$. Instead, for fixed $a_0 \in A_0$, 0 is a regular value of $f_{a_0} = f(\cdot, a_0)$ if and only if $D_x f(x, a_0)$ has full rank for every x such that $f_{a_0}(x) = f(x, a_0) = 0$.

Remark: Consider the important special case in which $n = m$, so we have as many equations (m) as endogenous variables (n). In this case, suppose f is C^1 (note that $1 = 1 + \max\{0, n - n\}$). If 0 is a regular value of f , that is, $Df(x, a)$ has rank $n = m$ for every (x, a) such that $f(x, a) = 0$, then by the Transversality Theorem there is a set $A_0 \subset A$ such that $A \setminus A_0$ has Lebesgue measure zero and for every $a_0 \in A_0$, $D_x f(x, a_0)$ has rank $n = m$ for all x such that $f(x, a_0) = 0$.

Fix $a_0 \in A_0$ and x_0 such that $f(x_0, a_0) = 0$. By the Implicit Function Theorem, there exist open sets A^* containing a_0 and X^* containing x_0 , and a C^1 function $x : A^* \rightarrow X^*$ such that

- $x(a_0) = x_0$

- $f(x(a), a) = 0$ for every $a \in A^*$
- if $(x, a) \in X^* \times A^*$ then

$$f(x, a) = 0 \iff x = x(a)$$

that is, x_0 is locally unique, and $x(a)$ is locally unique for each $a \in A^*$

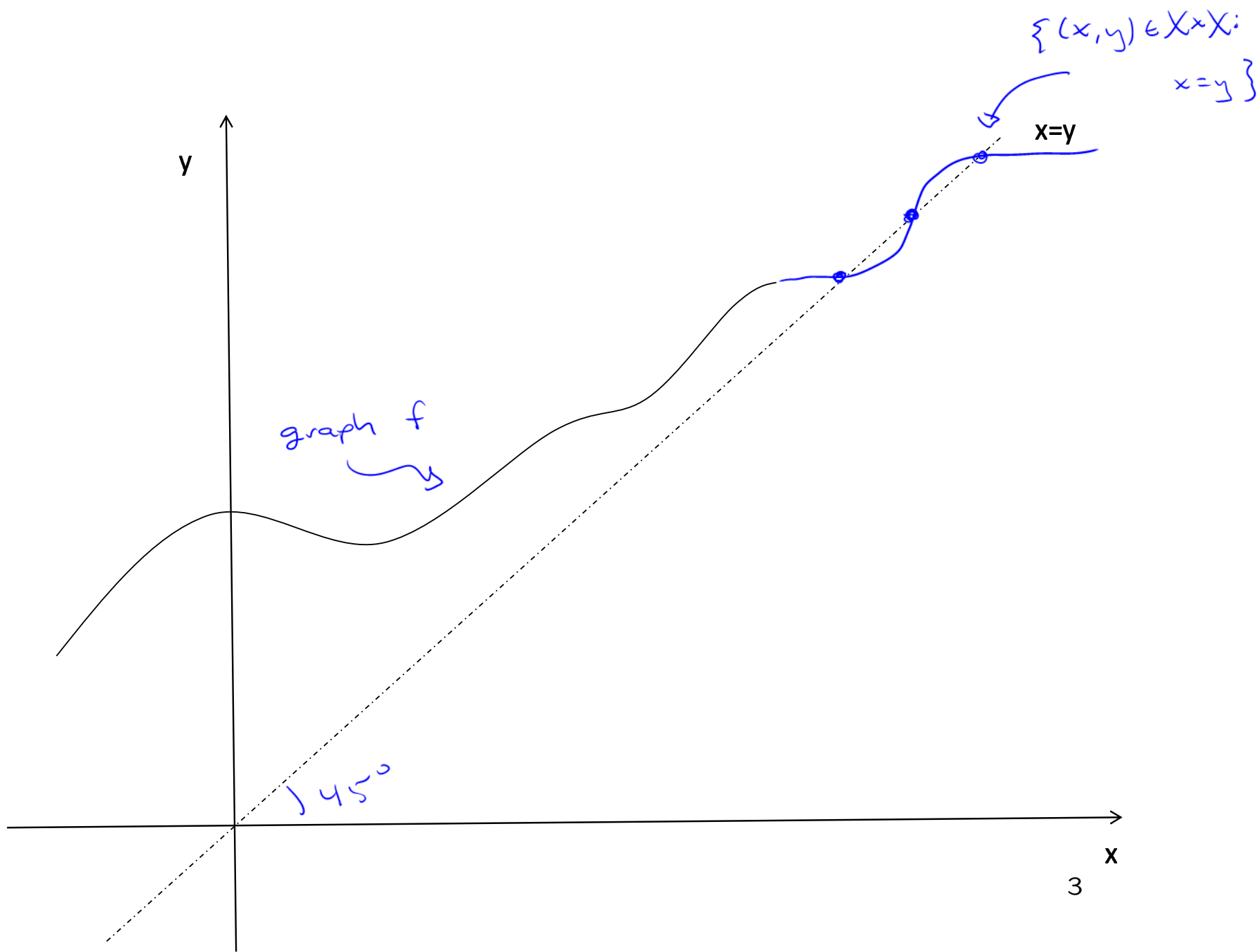
- $Dx(a_0) = -[D_x f(x_0, a_0)]^{-1} D_a f(x_0, a_0)$

Recall :

Fixed Points for Functions

Definition 1. Let X be a nonempty set and $f : X \rightarrow X$. A point $x^* \in X$ is a fixed point of f if $f(x^*) = x^*$.

x^* is a fixed point of f if it is “fixed” by the map f .



Fixed Points for Functions

Examples:

1. Let $X = \mathbf{R}$ and $f : \mathbf{R} \rightarrow \mathbf{R}$ be given by $f(x) = 2x$. Then $x = 0$ is a fixed point of f (and is the unique fixed point of f).

$$f(x) = 2x = x \iff x = 0$$

2. Let $X = \mathbf{R}$ and $f : \mathbf{R} \rightarrow \mathbf{R}$ be given by $f(x) = x$. Then every point in \mathbf{R} is a fixed point of f (in particular, fixed points need not be unique).

3. Let $X = \mathbf{R}$ and $f : \mathbf{R} \rightarrow \mathbf{R}$ be given by $f(x) = x + 1$. Then f has no fixed points.

$$f(x) = x + 1 \neq x \quad \forall x \in \mathbb{R}$$

4. Let $X = [0, 2]$ and $f : X \rightarrow X$ be given by $f(x) = \frac{1}{2}(x + 1)$.
Then

$$\begin{aligned} f(x) &= \frac{1}{2}(x + 1) = x \\ \iff x + 1 &= 2x \\ \iff x &= 1 \end{aligned}$$

So $x = 1$ is the unique fixed point of f . Notice that f is a contraction (why?), so we already knew that f must have a unique fixed point on \mathbf{R} from the Contraction Mapping Theorem.

5. Let $X = [0, \frac{1}{4}] \cup [\frac{3}{4}, 1]$ and $f : X \rightarrow X$ be given by $f(x) = 1 - x$.
Then f has no fixed points.

$$\begin{aligned} f(x) = 1 - x = x &\iff 1 = 2x \\ &\iff x = \frac{1}{2} \notin X \end{aligned}$$

6. Let $X = [-2, 2]$ and $f : X \rightarrow X$ be given by $f(x) = \frac{1}{2}x^2$. Then f has two fixed points, $x = 0$ and $x = 2$. If instead $X' = (0, 2)$, then $f : X' \rightarrow X'$ but f has no fixed points on X' .

7. Let $X = \{1, 2, 3\}$ and $f : X \rightarrow X$ be given by $f(1) = 2, f(2) = 3, f(3) = 1$ (so f is a permutation of X). Then f has no fixed points.

8. Let $X = [0, 2]$ and $f : X \rightarrow X$ be given by

$$f(x) = \begin{cases} x + 1 & \text{if } x \leq 1 \\ x - 1 & \text{if } x > 1 \end{cases}$$

Then f has no fixed points.

A Simple Fixed Point Theorem

Theorem 1. Let $X = [a, b]$ for $a, b \in \mathbf{R}$ with $a < b$ and let $f : X \rightarrow X$ be continuous. Then f has a fixed point.

Proof. Let $g : [a, b] \rightarrow \mathbf{R}$ be given by

$$g(x) = f(x) - x$$

$g(x) = 0 \Leftrightarrow x$ is a fixed point of f

If either $f(a) = a$ or $f(b) = b$, we're done. So assume $f(a) > a$ and $f(b) < b$. Then

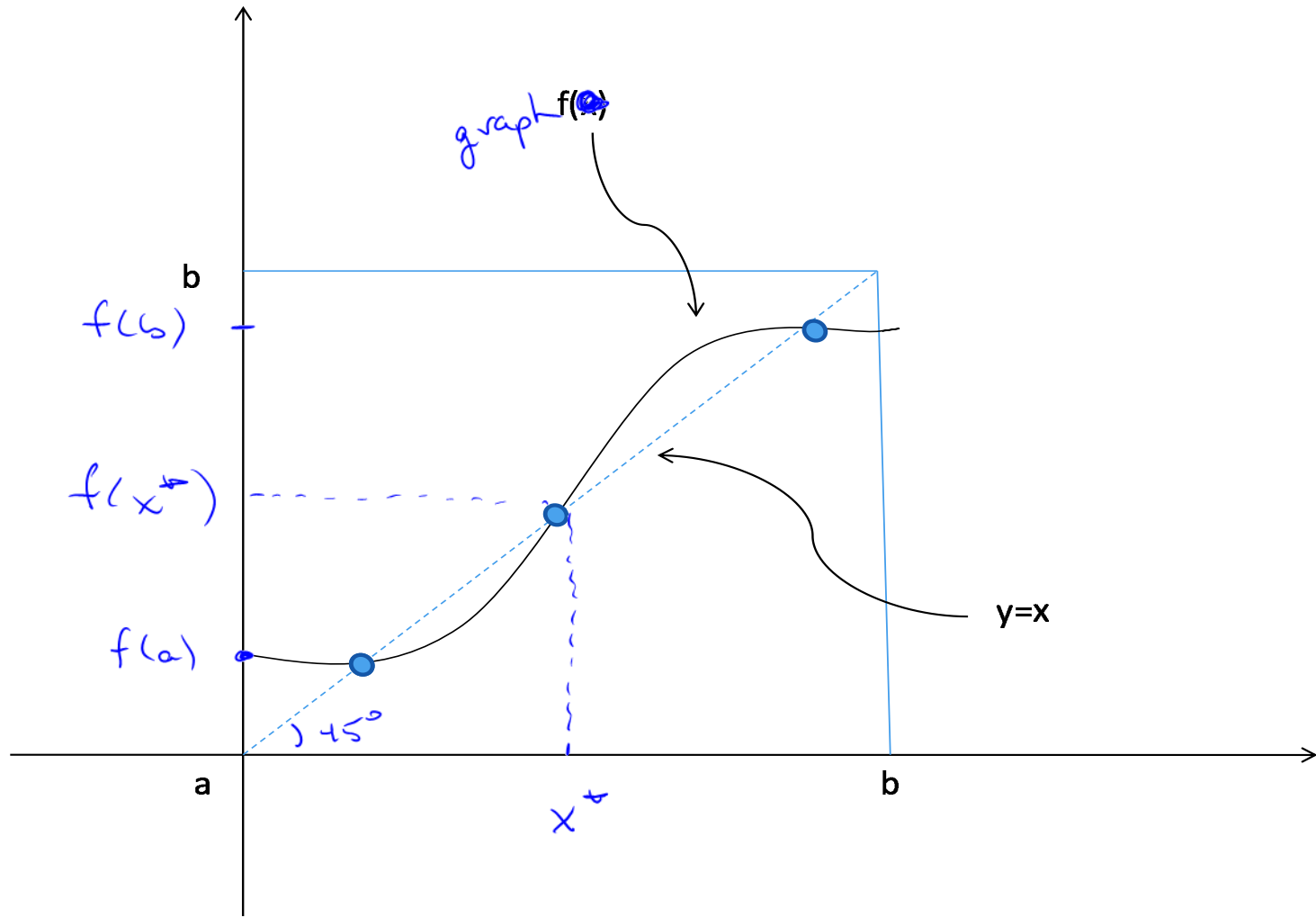
$(f(a) \in [a, b])$

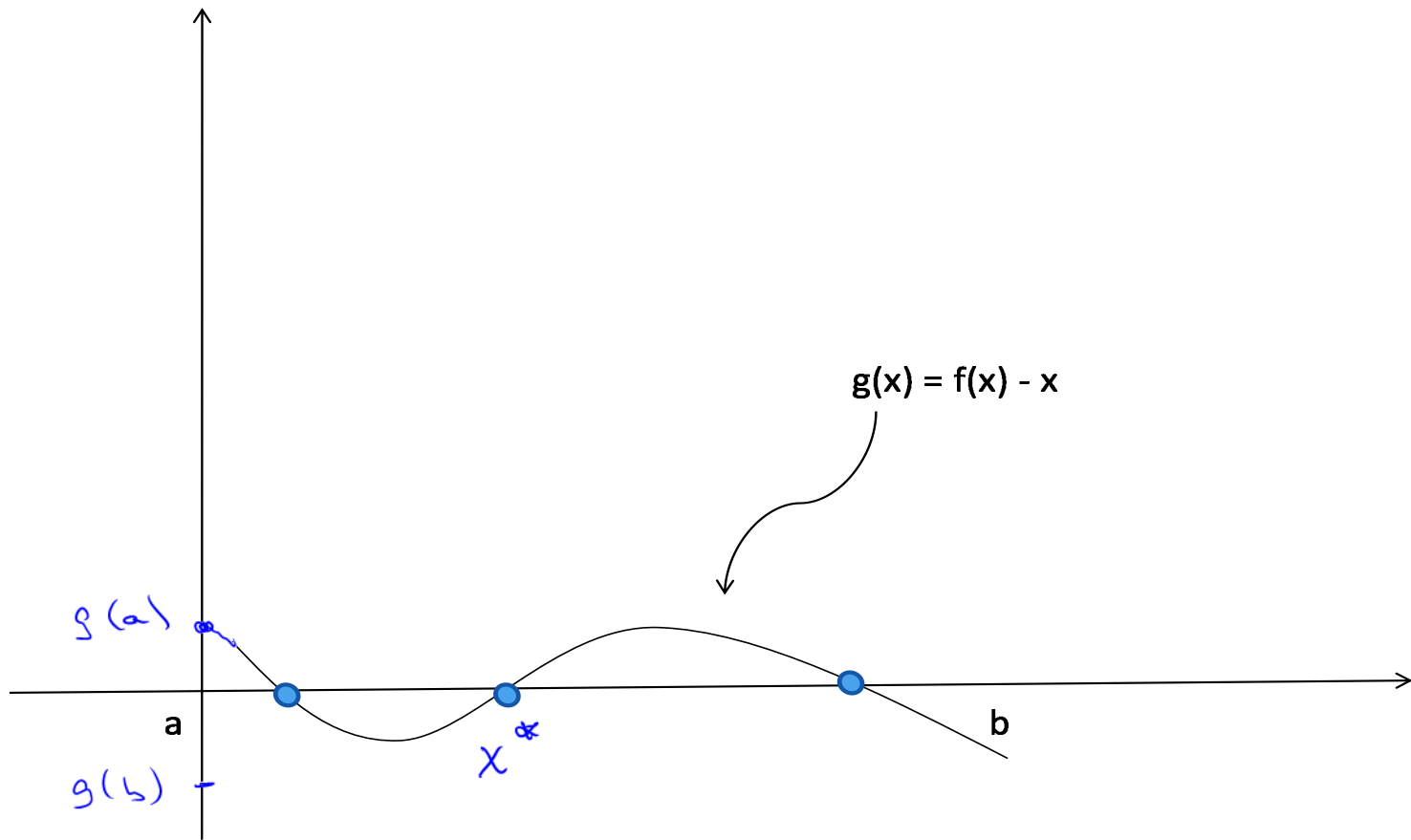
$(f(b) \in [a, b])$

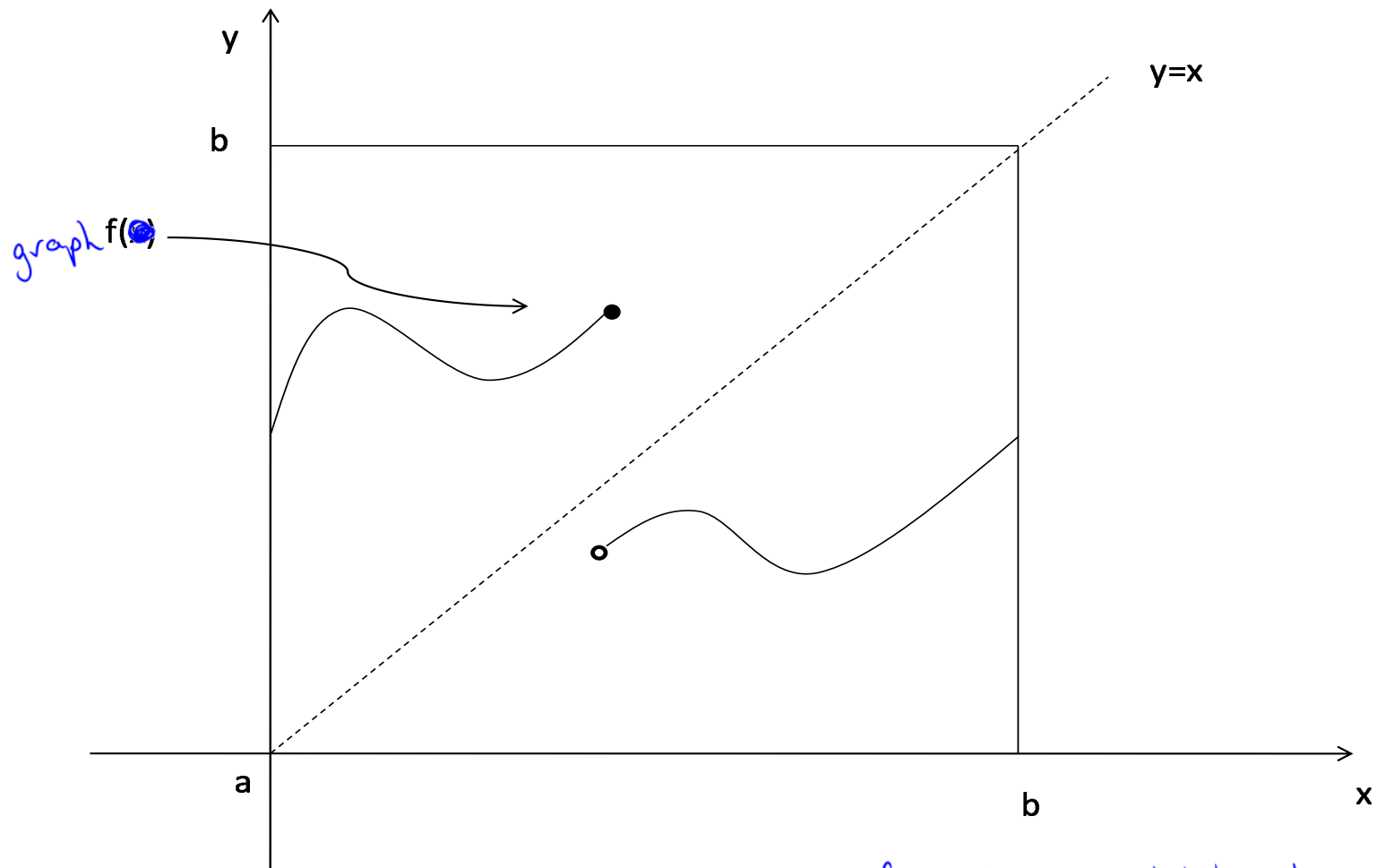
$$g(a) = f(a) - a > 0$$

$$g(b) = f(b) - b < 0$$

g is continuous, so by the Intermediate Value Theorem, $\exists x^* \in (a, b)$ such that $g(x^*) = 0$, that is, such that $f(x^*) = x^*$. \square







a discontinuous function might have
no fixed points

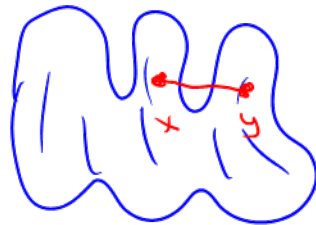
Brouwer's Fixed Point Theorem

Theorem 2 (Thm. 3.2. Brouwer's Fixed Point Theorem). Let $X \subseteq \mathbb{R}^n$ be nonempty, compact, and convex, and let $f : X \rightarrow X$ be continuous. Then f has a fixed point.

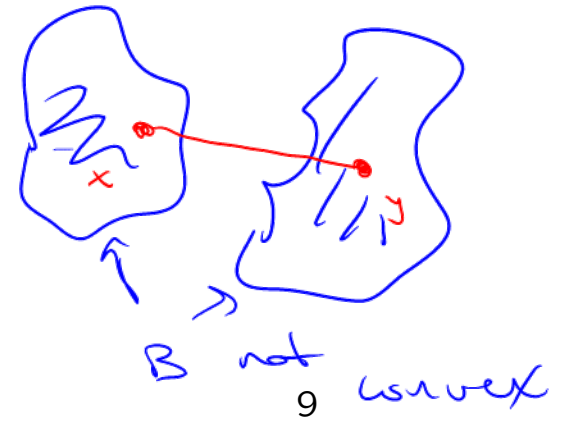
$X \subseteq \mathbb{R}^n$ is convex if $\forall x, y \in X \quad \forall \alpha \in [0, 1]$
 $\alpha x + (1 - \alpha)y \in X$



X convex



D not convex



B not convex

X non empty, compact, convex \Rightarrow
 X homeomorphic to B^m $\exists h: X \rightarrow B^m$ 1-1,

Sketch of Proof of Brouwer

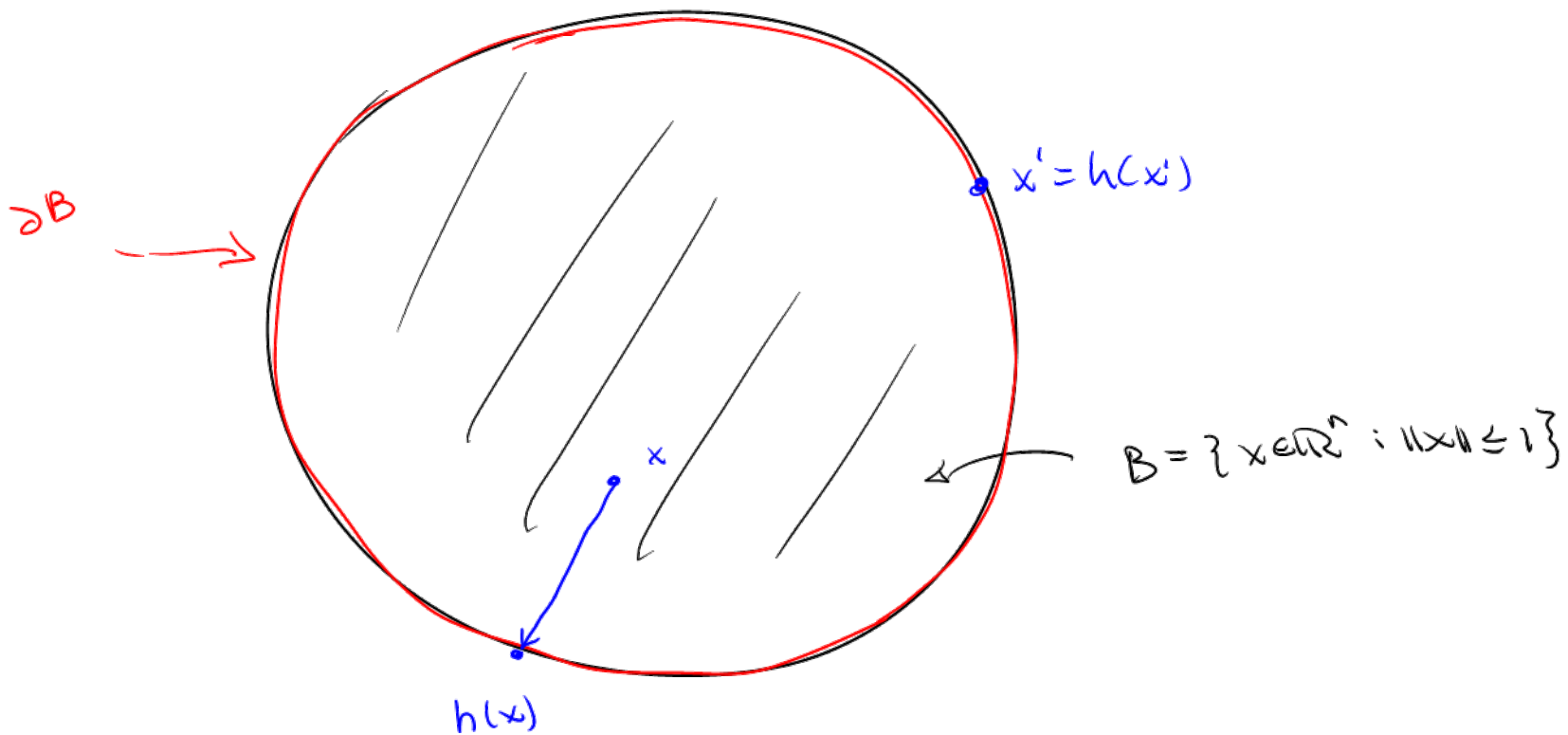
onto, cont.
 h^{-1} cont.

Consider the case when the set X is the ^{closed} unit ball in \mathbf{R}^n , i.e. $X = B_1[0] = B = \{x \in \mathbf{R}^n : \|x\| \leq 1\}$. Let $f: B \rightarrow B$ be a continuous function. Recall that ∂B denotes the boundary of B , so $\partial B = \{x \in \mathbf{R}^n : \|x\| = 1\}$.

Fact: Let B be the unit ball in \mathbf{R}^n . Then there is no continuous function $h: B \rightarrow \partial B$ such that $h(x') = x'$ for every $x' \in \partial B$.

See J. Franklin, Methods of Mathematical Economics, for an elementary (but long) proof.

(also Y. Kannai, Am. Math. Monthly, April 1981, pp 264-268)



$\exists h: B \rightarrow \partial B$ continuous such that
 $x' = h(x) \quad \forall x' \in \partial B$

Now to establish Brouwer's theorem, suppose, by way of contradiction, that f has no fixed points in B . Thus for every $x \in B$, $x \neq f(x)$.

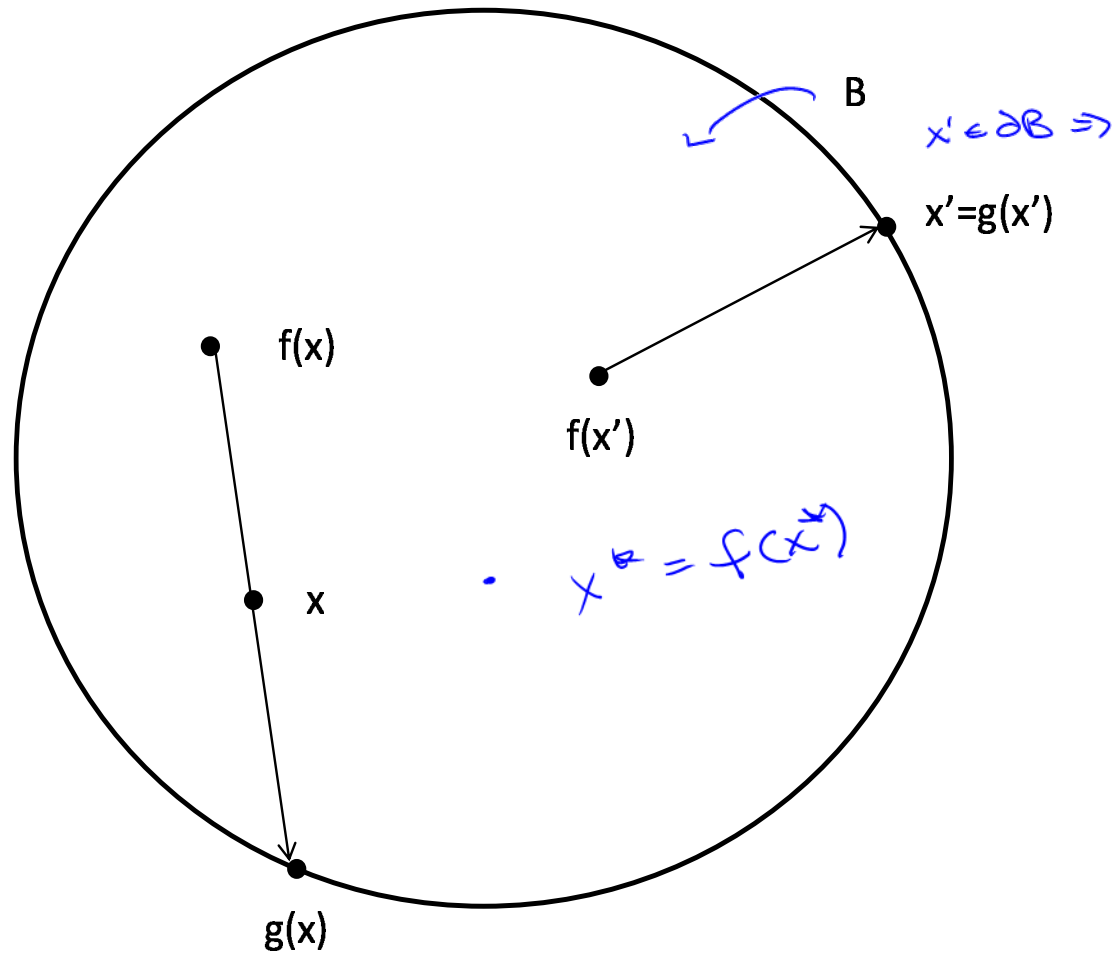
Since $x \neq f(x)$ for every x , we can carry out the following construction. For each $x \in B$, construct the line segment originating at $f(x)$ and going through x . Let $g(x)$ denote the intersection of this line segment with ∂B .

This construction is well-defined, and gives a continuous function $g : B \rightarrow \partial B$. Furthermore, if $x' \in \partial B$, then $x' = g(x')$. That is, $g|_{\partial B} = \text{id}_{\partial B}$. Since there are no such functions by the fact above, we have a contradiction. Therefore there exists $x^* \in B$ such that $f(x^*) = x^*$, that is, f has a fixed point in B .

$$g(x) = x + tu$$

where $u = \frac{x - f(x)}{\|x - f(x)\|}$

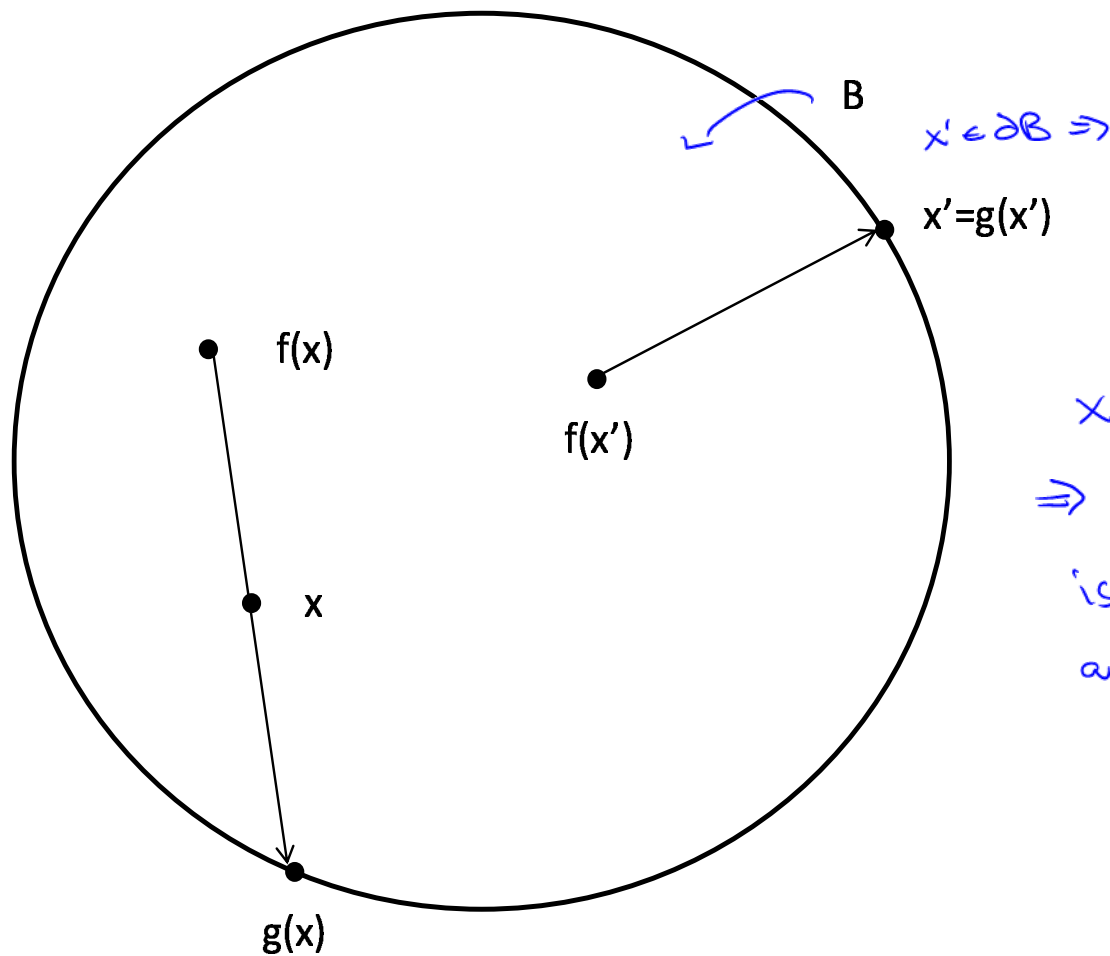
$$t = -x \cdot u + \sqrt{1 - x \cdot x + (x \cdot u)^2}$$



$$g(x) = x + tu$$

where $u = \frac{x - f(x)}{\|x - f(x)\|}$

$$t = -x \cdot u + \sqrt{1 - x \cdot x + (x \cdot u)^2}$$



$x \neq f(x) \forall x$
 $\Rightarrow g: B \rightarrow \partial B$
 is well-defined
 and continuous
 ~~$\Rightarrow \chi =$~~

Fixed Points for Correspondences

Definition 2. Let X be nonempty and $\Psi : X \rightarrow 2^X$ be a correspondence. A point $x^* \in X$ is a fixed point of Ψ if $x^* \in \Psi(x^*)$.

Note here that we do *not* require $\Psi(x^*) = \{x^*\}$, that is Ψ need not be single-valued at x^* . So x^* can be a fixed point of Ψ but there may be other elements of $\Psi(x^*)$ different from x^* .

Examples:

1. Let $X = [0, 4]$ and $\psi : X \rightarrow 2^X$ be given by

$$\psi(x) = \begin{cases} [x + 1, x + 2] & \text{if } x < 2 \\ [0, 4] & \text{if } x = 2 \\ [x - 2, x - 1] & \text{if } x > 2 \end{cases}$$

Then $x = 2$ is the unique fixed point of ψ .

$$2 \in \psi(2) = [0, 4]$$

$$x \notin [x+1, x+2]$$

$$x \notin [x-2, x-1]$$

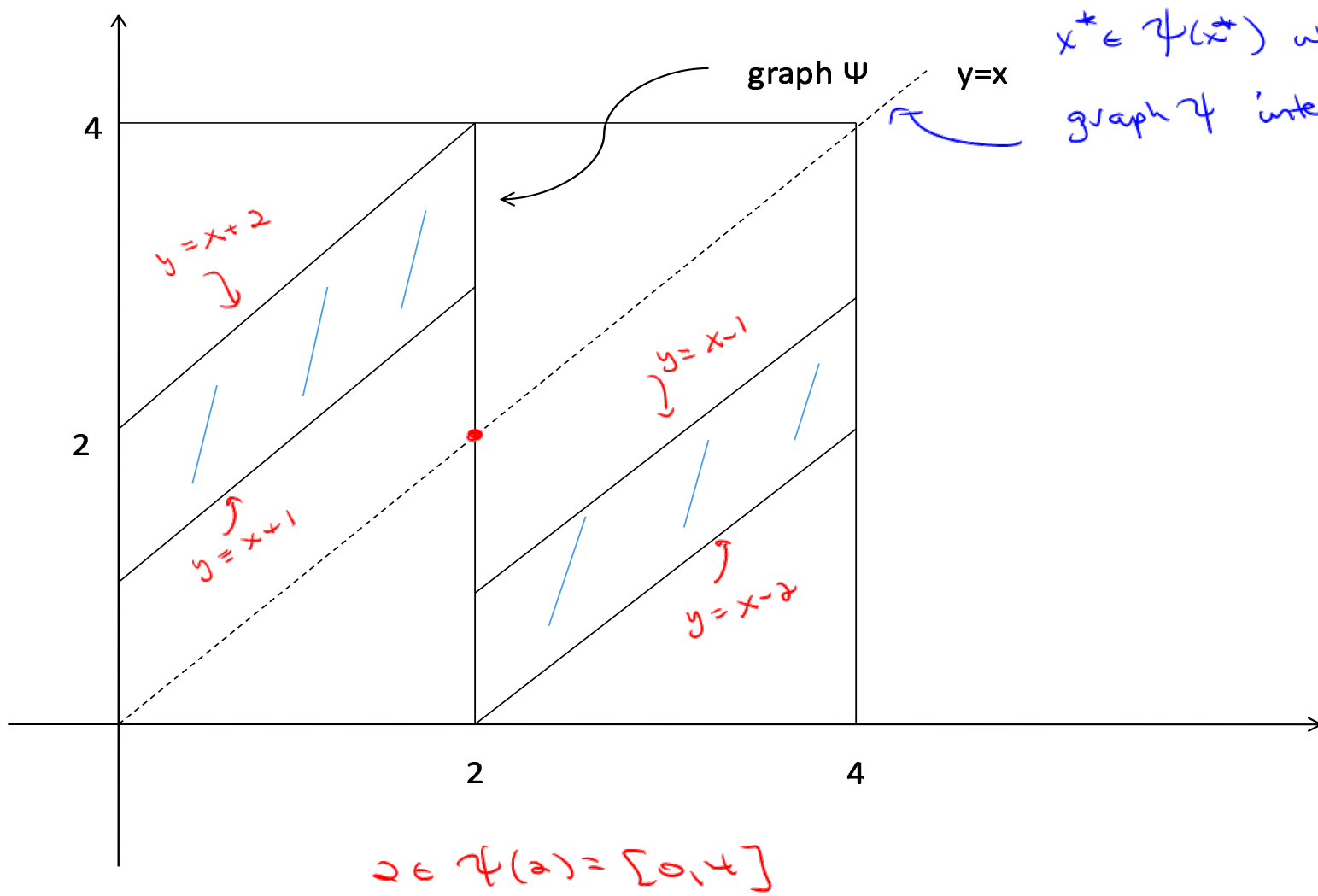
$$\Rightarrow x \notin \psi(x) \\ \forall x \neq 2$$

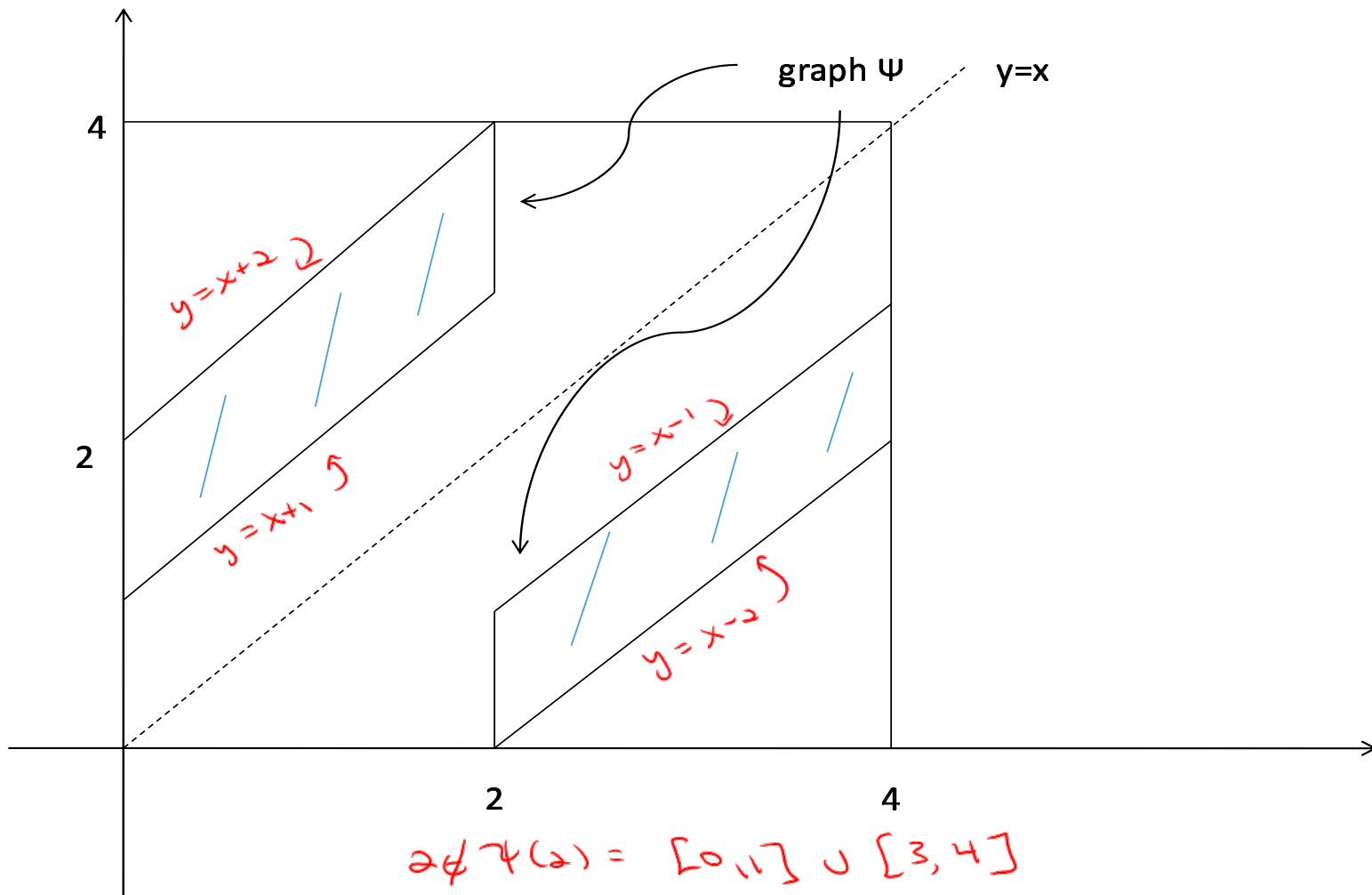
2. Let $X = [0, 4]$ and $\psi : X \rightarrow 2^X$ be given by

$$\psi(x) = \begin{cases} [x + 1, x + 2] & \text{if } x < 2 \\ [0, 1] \cup [3, 4] & \text{if } x = 2 \\ [x - 2, x - 1] & \text{if } x > 2 \end{cases}$$

Then ψ has no fixed points.

$$2 \notin \psi(2) = [0, 1] \cup [3, 4]$$





Note: ψ is onto in both cases

$\psi(x) \subseteq \mathbb{R}^n$ is nonempty, compact, convex set
 $\forall x \in X$

Kakutani's Fixed Point Theorem

Theorem 3. (Thm. 3.4'. Kakutani's Fixed Point Theorem)

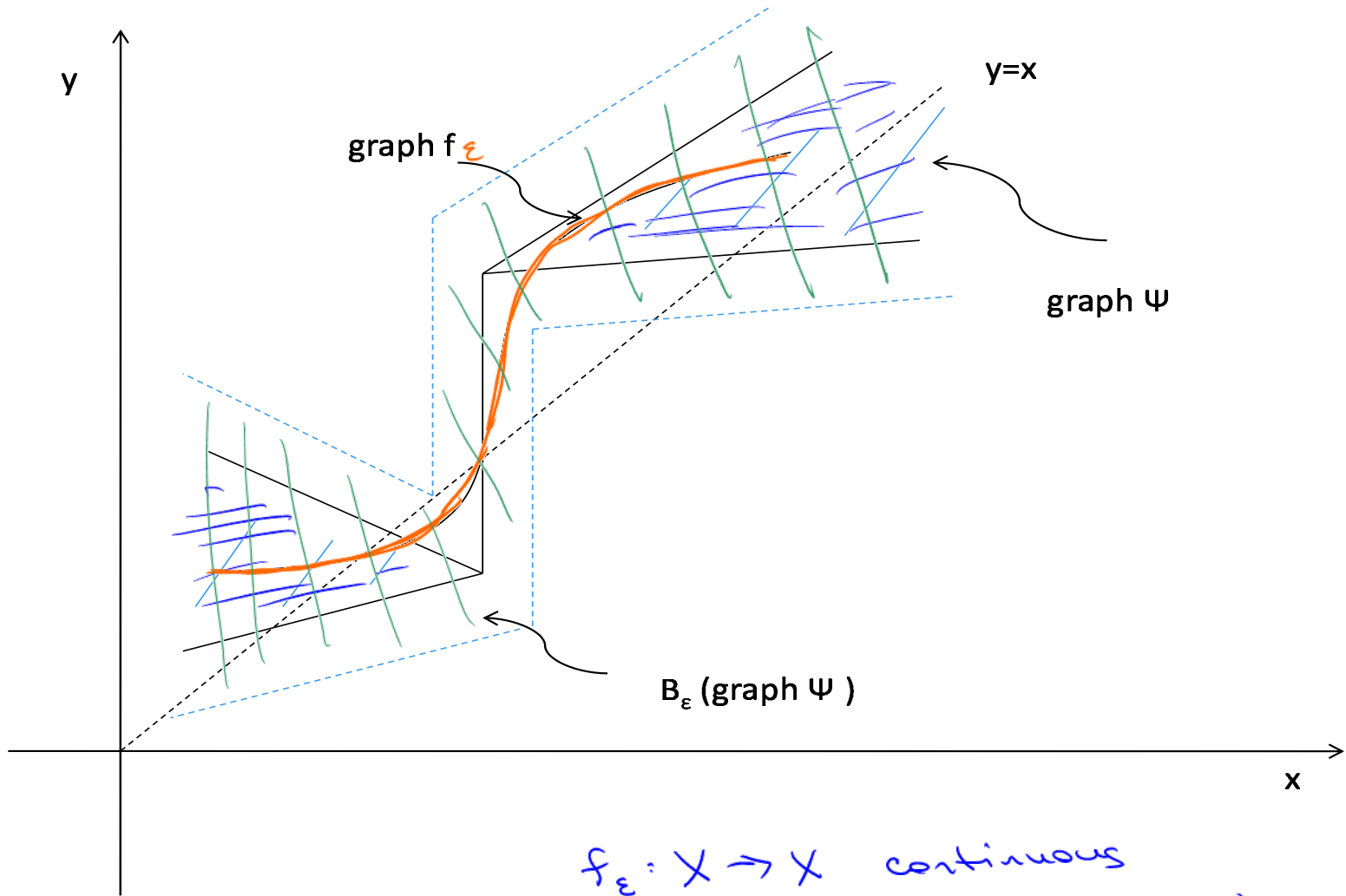
Let $X \subseteq \mathbb{R}^n$ be a non-empty, compact, convex set and $\Psi : X \rightarrow 2^X$ be an ~~upper~~ ^{lower} hemi-continuous correspondence with non-empty, convex, compact values. Then Ψ has a fixed point in X .

Proof. (sketch) Here, the idea is to use Brouwer's theorem after appropriately approximating the correspondence with a function. ~~The catch is that there won't necessarily exist~~ ^S a continuous selection from Ψ , that is, a continuous function $f : X \rightarrow X$ such that $f(x) \in \Psi(x)$ for every $x \in X$. If such a function existed, then by applying Brouwer to f we would have a fixed point of Ψ (because if $\exists x^* \in X$ such that $x^* = f(x^*)$, then $x^* = f(x^*) \in \Psi(x^*)$).

Instead, we look for a weaker type of approximation. Let $X \subset \mathbf{R}^n$ be a non-empty, compact, convex set, and let $\Psi : X \rightarrow 2^X$ be an uhc correspondence with non-empty, compact, convex values. For every $\varepsilon > 0$, define the ε ball about graph Ψ to be

$$B_\varepsilon(\text{graph } \Psi) = \left\{ z \in X \times X : d(z, \text{graph } \Psi) = \inf_{(x,y) \in \text{graph } \Psi} d(z, (x,y)) < \varepsilon \right\}$$

Here d denotes the ordinary Euclidean distance. Since Ψ is a convex-valued correspondence, for every $\varepsilon > 0$ there exists a continuous function $f_\varepsilon : X \rightarrow X$ such that $\text{graph } f_\varepsilon \subseteq B_\varepsilon(\text{graph } \Psi)$.



$f_\epsilon: X \rightarrow X$ continuous
 $\text{graph } f_\epsilon \subseteq B_\epsilon(\text{graph } \psi)$

$$\varepsilon_n = \frac{1}{n} \quad \psi_n \quad f_n: X \rightarrow X \quad \psi_n$$

Now by letting $\varepsilon \rightarrow 0$, this means that we can find a sequence of continuous functions $\{f_n\}$ such that $\text{graph } f_n \subseteq B_{\frac{1}{n}}(\text{graph } \psi)$ for each n . By Brouwer's Fixed Point Theorem, each function f_n has a fixed point $\hat{x}_n \in X$, and

$$(\hat{x}_n, \hat{x}_n) = (\hat{x}_n, f_n(\hat{x}_n)) \in \text{graph } f_n \subseteq B_{\frac{1}{n}}(\text{graph } \psi) \text{ for each } n$$

So for each n there exists $(x_n, y_n) \in \text{graph } \psi$ such that $d((\hat{x}_n, \hat{x}_n), (x_n, y_n)) < \frac{1}{n}$

$$d(\hat{x}_n, x_n) < \frac{1}{n} \text{ and } d(\hat{x}_n, y_n) < \frac{1}{n}$$

Since X is compact, $\{\hat{x}_n\}$ has a convergent subsequence $\{\hat{x}_{n_k}\}$, with $\hat{x}_{n_k} \rightarrow \hat{x} \in X$. Then $x_{n_k} \rightarrow \hat{x}$ and $y_{n_k} \rightarrow \hat{x}$. Since ψ is usc and closed-valued, it has closed graph, so $(\hat{x}, \hat{x}) \in \text{graph } \psi$. Thus $\hat{x} \in \psi(\hat{x})$, that is, \hat{x} is a fixed point of ψ . \square

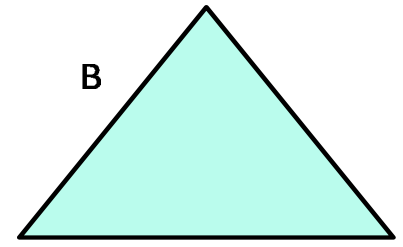
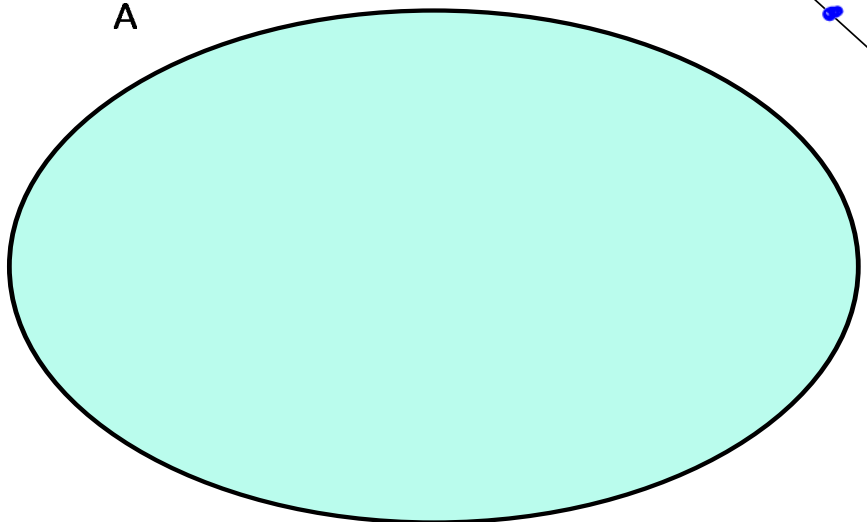
Separating Hyperplane Theorems

Theorem 4 (1.26, Separating Hyperplane Theorem). *Let $A, B \subseteq \mathbb{R}^n$ be nonempty, disjoint convex sets. Then there exists a nonzero vector $p \in \mathbb{R}^n$ such that*

$$p \cdot a \leq p \cdot b \quad \forall a \in A, b \in B$$

hyperplane: $\{z \in \mathbb{R}^n : v \cdot x = c\}$ for some
 $v \neq 0, v \in \mathbb{R}^n$
some $c \in \mathbb{R}$

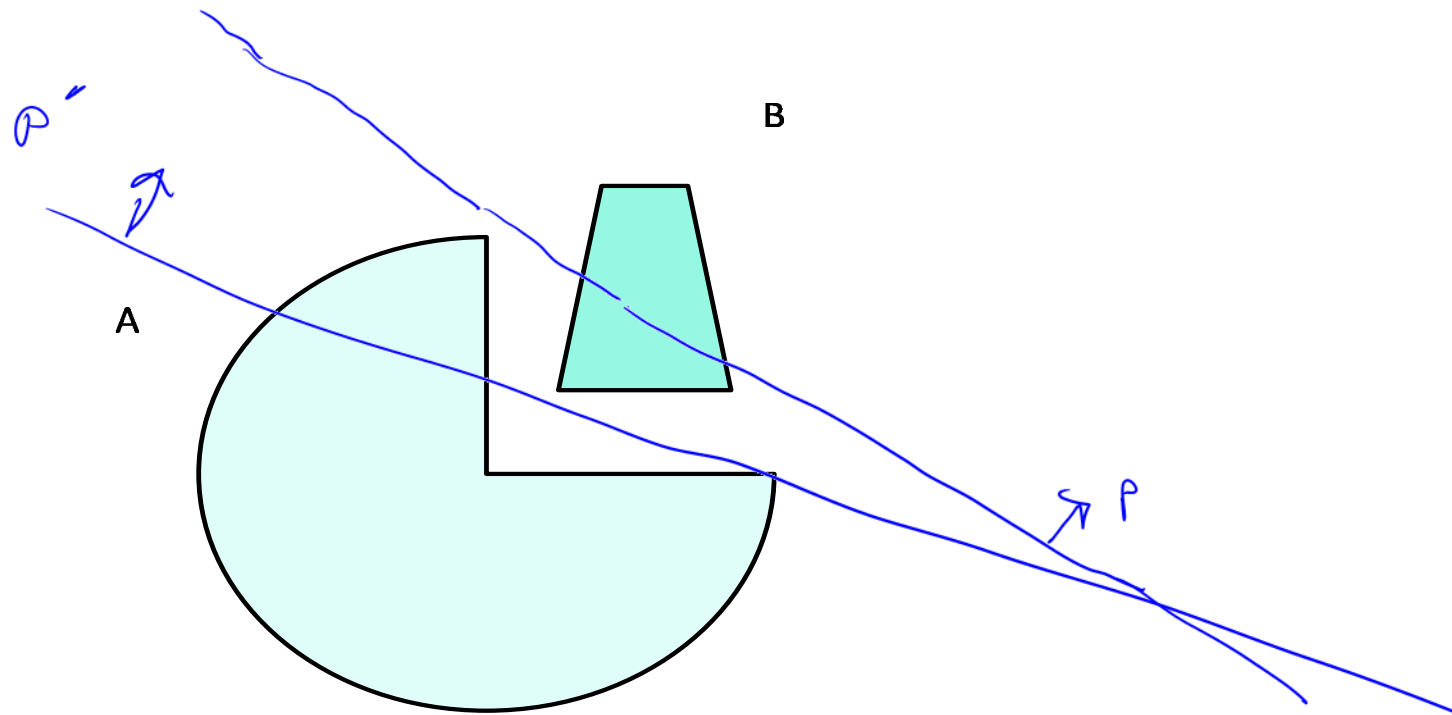
$$H = \{z \in \mathbb{R}^n : p \cdot z = c\}$$



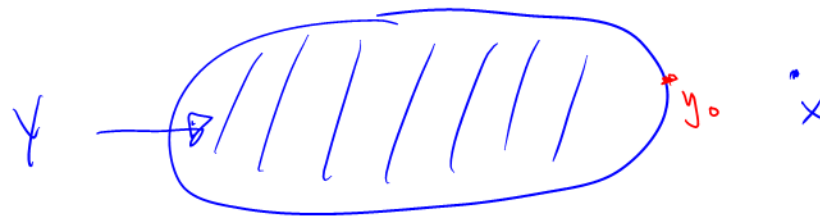
$$p \cdot a \leq c \\ \forall a \in A$$

$$p \cdot b \geq c \\ \forall b \in B$$

$$c = p \cdot \bar{z} \\ \exists \bar{z} : p \cdot \bar{z} = p \cdot \bar{z}$$



convexity important : no hyperplane separates
A and B



Separating a Point from a Set

Theorem 5. Let $Y \subseteq \mathbf{R}^n$ be a nonempty convex set and $x \notin Y$. Then there exists a nonzero vector $p \in \mathbf{R}^n$ such that

$$p \cdot x \leq p \cdot y \quad \forall y \in Y$$

Proof. We sketch the proof in the special case that Y is compact. We will see that in this case we actually get a stronger conclusion:

$$\exists p \in \mathbf{R}^n, p \neq 0 \text{ s.t. } p \cdot x < p \cdot y \quad \forall y \in Y$$

Choose $y_0 \in Y$ such that $\|y_0 - x\| = \inf\{\|y - x\| : y \in Y\}$; such a point exists because Y is compact, so the distance function $g(y) = \|y - x\|$ assumes its minimum on Y . Since $x \notin Y$, $x \neq y_0$, so $y_0 - x \neq 0$. Let $p = y_0 - x$. The set

$$H = \{z \in \mathbf{R}^n : p \cdot z = p \cdot y_0\}$$

*g is continuous
(PS 3)*

is the hyperplane perpendicular to p through y_0 . See Figure 12.
Then

$$\begin{aligned} p \cdot y_0 &= (y_0 - x) \cdot y_0 \\ &= (y_0 - x) \cdot (y_0 - x + x) \\ &= (y_0 - x) \cdot (y_0 - x) + (y_0 - x) \cdot x \\ &= \|y_0 - x\|^2 + p \cdot x \\ &> p \cdot x \end{aligned}$$

We claim that

$$y \in Y \Rightarrow p \cdot y \geq p \cdot y_0 > p \cdot x$$

If not, suppose there exists $y \in Y$ such that $p \cdot y < p \cdot y_0$. Given $\alpha \in (0, 1)$, let

$$w_\alpha = \alpha y + (1 - \alpha)y_0$$

Since Y is convex, $w_\alpha \in Y$. Then for α sufficiently close to zero,

$$\begin{aligned}
 \|x - w_\alpha\|^2 &= \|x - \alpha y - (1 - \alpha)y_0\|^2 && \text{defn of } w_\alpha \\
 &= \|x - y_0 + \alpha(y_0 - y)\|^2 && \text{algebra} \\
 &= \|-p + \alpha(y_0 - y)\|^2 && \text{defn of } p \\
 &= |p|^2 - 2\alpha p \cdot (y_0 - y) + \alpha^2 |y_0 - y|^2 && \text{more algebra} \\
 &= |p|^2 + \alpha \left(\underbrace{-2p \cdot (y_0 - y)}_{\text{negative}} + \alpha |y_0 - y|^2 \right) && \text{"} \\
 &< |p|^2 \quad \text{for } \alpha \text{ close to 0, as } p \cdot y_0 > p \cdot y \rightarrow 0 \text{ as } \alpha \rightarrow 0 && \text{red} \\
 &= \|y_0 - x\|^2
 \end{aligned}$$

Thus for α sufficiently close to zero,

$$\|w_\alpha - x\| < \|y_0 - x\|$$

which implies y_0 is not the closest point in Y to x , contradiction.

□

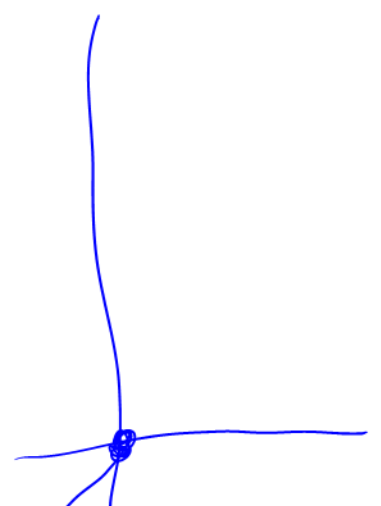
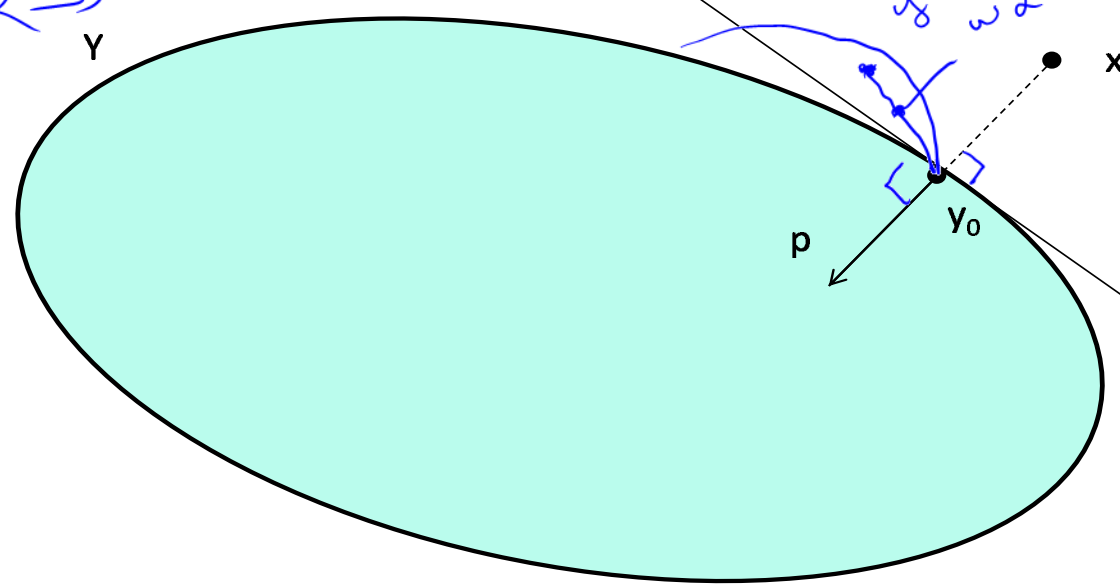
$$\cos \Theta = \frac{v \cdot w}{\|v\| \|w\|}$$



$$v \cdot w \geq 0 \iff \Theta \leq 90^\circ$$

$$v \cdot w \leq 0 \iff \Theta \geq 90^\circ$$

$$H = \{z : p \cdot z = p \cdot y_0\}$$



$$p \cdot y_0 = (y_0 - x) \cdot y_0$$

$$= y_0 \cdot y_0 - x \cdot y_0$$

$$p \cdot y \geq p \cdot y_0 \quad \forall y \in Y$$

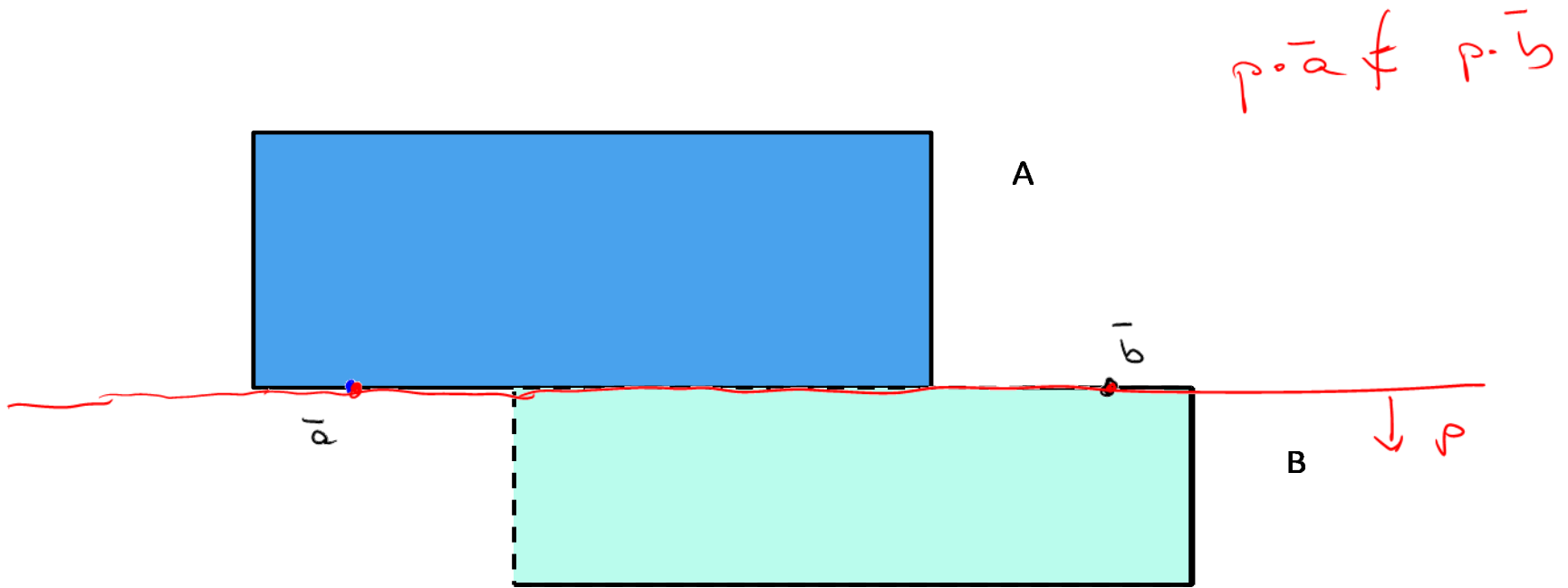
The general version of the Separating Hyperplane Theorem can be derived from this special case by noting that if $A \cap B = \emptyset$, then $0 \notin A - B = \{a - b : a \in A, b \in B\}$.

Strict Separation

For the special case of Y compact and $X = \{x\}$, we actually could *strictly separate* Y and X :

$$x \notin Y \Rightarrow \exists p \in \mathbf{R}^n, p \neq 0 \text{ s.t. } p \cdot x < p \cdot y \quad \forall y \in Y$$

When can we do this in general? Will require additional assumptions...



A, B nonempty, disjoint, convex \Rightarrow

$$\exists p \in \mathbb{R}^n, p \neq 0 \text{ st. } p \cdot a \leq p \cdot b \quad \forall a \in A, \forall b \in B$$

But

$$p \cdot \bar{a} = p \cdot \bar{b} \text{ for some } \bar{a} \in A \text{ and } \bar{b} \in B$$

(for any such p)

Strict Separation

Theorem 6. (Strict Separating Hyperplane Theorem) *Let $A, B \subseteq \mathbf{R}^n$ be nonempty, disjoint, closed, convex sets. Then there exists a nonzero vector $p \in \mathbf{R}^n$ such that*

$$p \cdot a < p \cdot b \quad \forall a \in A, b \in B$$

$$h = y - x$$

$$0 < 2m + f'(x)h + \frac{1}{2}Nh^2$$

$$g(h) = \frac{1}{2}Nh^2 + f'(x)h + 2m > 0 \quad \forall h$$

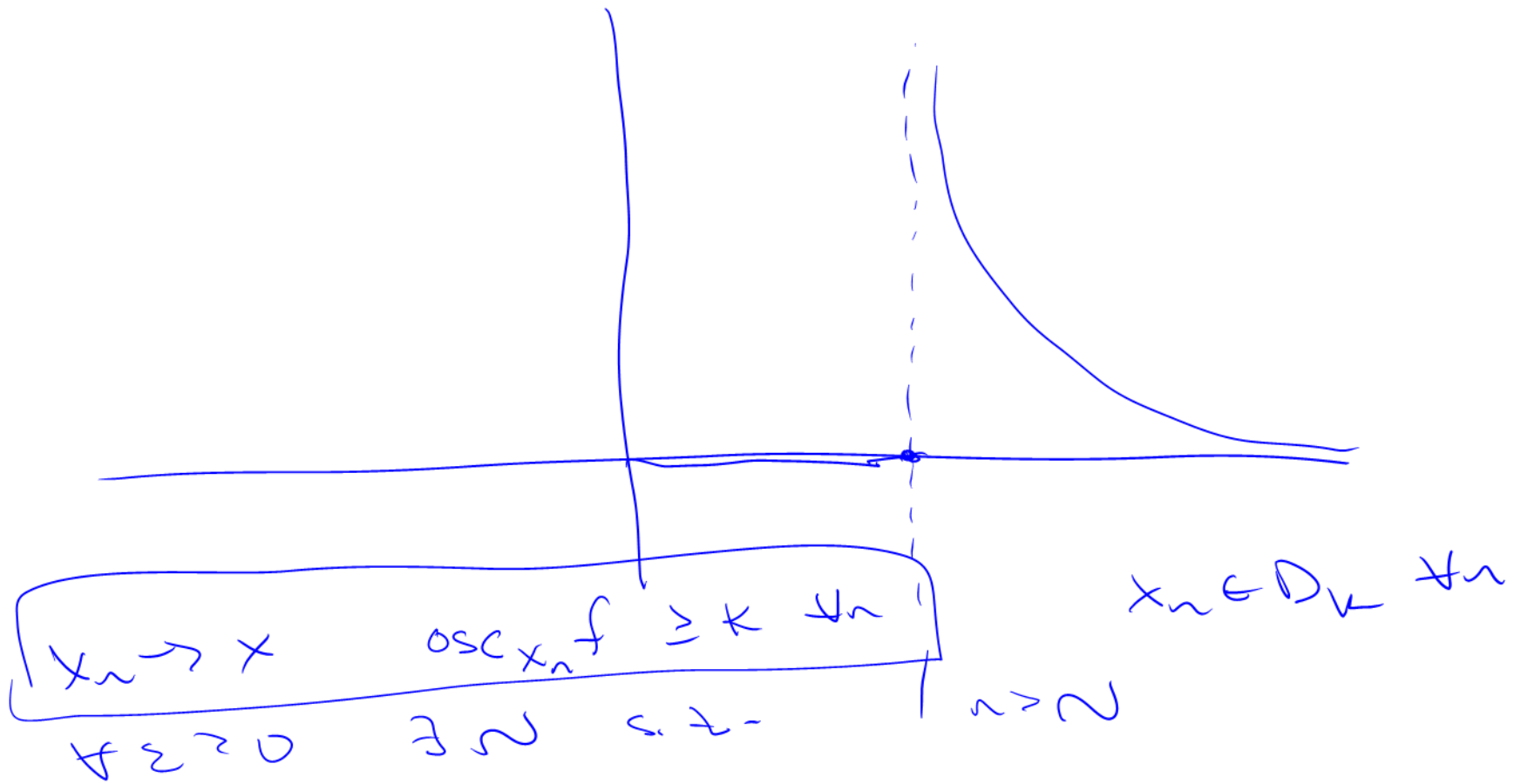
$$\Rightarrow (f'(x))^2 - 4\left(2m \frac{1}{2}N\right) < 0$$

$$\Rightarrow f'(x)^2 - 4mN < 0$$

||

$$f'(x)h \leq 2m + \frac{1}{2}Nh^2$$

$$f'(x) \leq \frac{2m}{|h|} + \frac{1}{2}N|h| \quad h = 2\sqrt{m/N}$$



$$x_n \in (x - \varepsilon/2, x + \varepsilon/2)$$

$$f([x - \varepsilon, x + \varepsilon]) \supseteq f([x_n - \varepsilon/2, x_n + \varepsilon/2])$$

$$\Rightarrow \text{diam } f([x - \varepsilon, x + \varepsilon]) \geq \text{diam } f([x_n - \varepsilon/2, x_n + \varepsilon/2]) \geq K$$

