

# Econ 204 2020

## Lecture 13

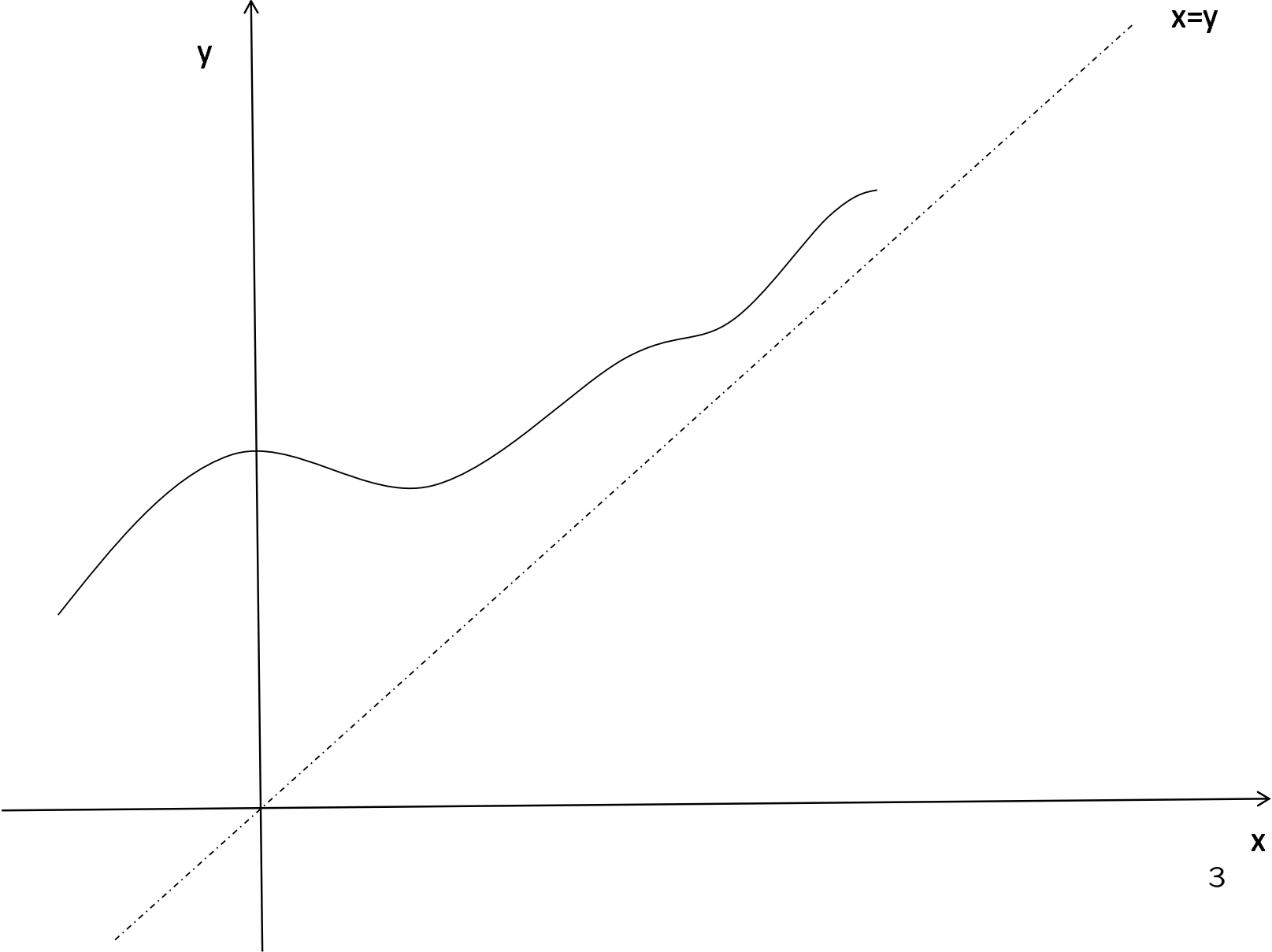
### Outline

1. Fixed Points for Functions
2. Brouwer's Fixed Point Theorem
3. Fixed Points for Correspondences
4. Kakutani's Fixed Point Theorem
5. Separating Hyperplane Theorems

## Fixed Points for Functions

**Definition 1.** *Let  $X$  be a nonempty set and  $f : X \rightarrow X$ . A point  $x^* \in X$  is a fixed point of  $f$  if  $f(x^*) = x^*$ .*

$x^*$  is a fixed point of  $f$  if it is “fixed” by the map  $f$ .



# Fixed Points for Functions

## Examples:

1. Let  $X = \mathbf{R}$  and  $f : \mathbf{R} \rightarrow \mathbf{R}$  be given by  $f(x) = 2x$ . Then  $x = 0$  is a fixed point of  $f$  (and is the unique fixed point of  $f$ ).
2. Let  $X = \mathbf{R}$  and  $f : \mathbf{R} \rightarrow \mathbf{R}$  be given by  $f(x) = x$ . Then every point in  $\mathbf{R}$  is a fixed point of  $f$  (in particular, fixed points need not be unique).
3. Let  $X = \mathbf{R}$  and  $f : \mathbf{R} \rightarrow \mathbf{R}$  be given by  $f(x) = x + 1$ . Then  $f$  has no fixed points.

4. Let  $X = [0, 2]$  and  $f : X \rightarrow X$  be given by  $f(x) = \frac{1}{2}(x + 1)$ .  
Then

$$\begin{aligned} f(x) &= \frac{1}{2}(x + 1) = x \\ &\iff x + 1 = 2x \\ &\iff x = 1 \end{aligned}$$

So  $x = 1$  is the unique fixed point of  $f$ . Notice that  $f$  is a contraction (why?), so we already knew that  $f$  must have a unique fixed point on  $\mathbf{R}$  from the Contraction Mapping Theorem.

5. Let  $X = [0, \frac{1}{4}] \cup [\frac{3}{4}, 1]$  and  $f : X \rightarrow X$  be given by  $f(x) = 1 - x$ .  
Then  $f$  has no fixed points.

6. Let  $X = [-2, 2]$  and  $f : X \rightarrow X$  be given by  $f(x) = \frac{1}{2}x^2$ . Then  $f$  has two fixed points,  $x = 0$  and  $x = 2$ . If instead  $X' = (0, 2)$ , then  $f : X' \rightarrow X'$  but  $f$  has no fixed points on  $X'$ .

7. Let  $X = \{1, 2, 3\}$  and  $f : X \rightarrow X$  be given by  $f(1) = 2, f(2) = 3, f(3) = 1$  (so  $f$  is a permutation of  $X$ ). Then  $f$  has no fixed points.

8. Let  $X = [0, 2]$  and  $f : X \rightarrow X$  be given by

$$f(x) = \begin{cases} x + 1 & \text{if } x \leq 1 \\ x - 1 & \text{if } x > 1 \end{cases}$$

Then  $f$  has no fixed points.

# A Simple Fixed Point Theorem

**Theorem 1.** *Let  $X = [a, b]$  for  $a, b \in \mathbf{R}$  with  $a < b$  and let  $f : X \rightarrow X$  be continuous. Then  $f$  has a fixed point.*

*Proof.* Let  $g : [a, b] \rightarrow \mathbf{R}$  be given by

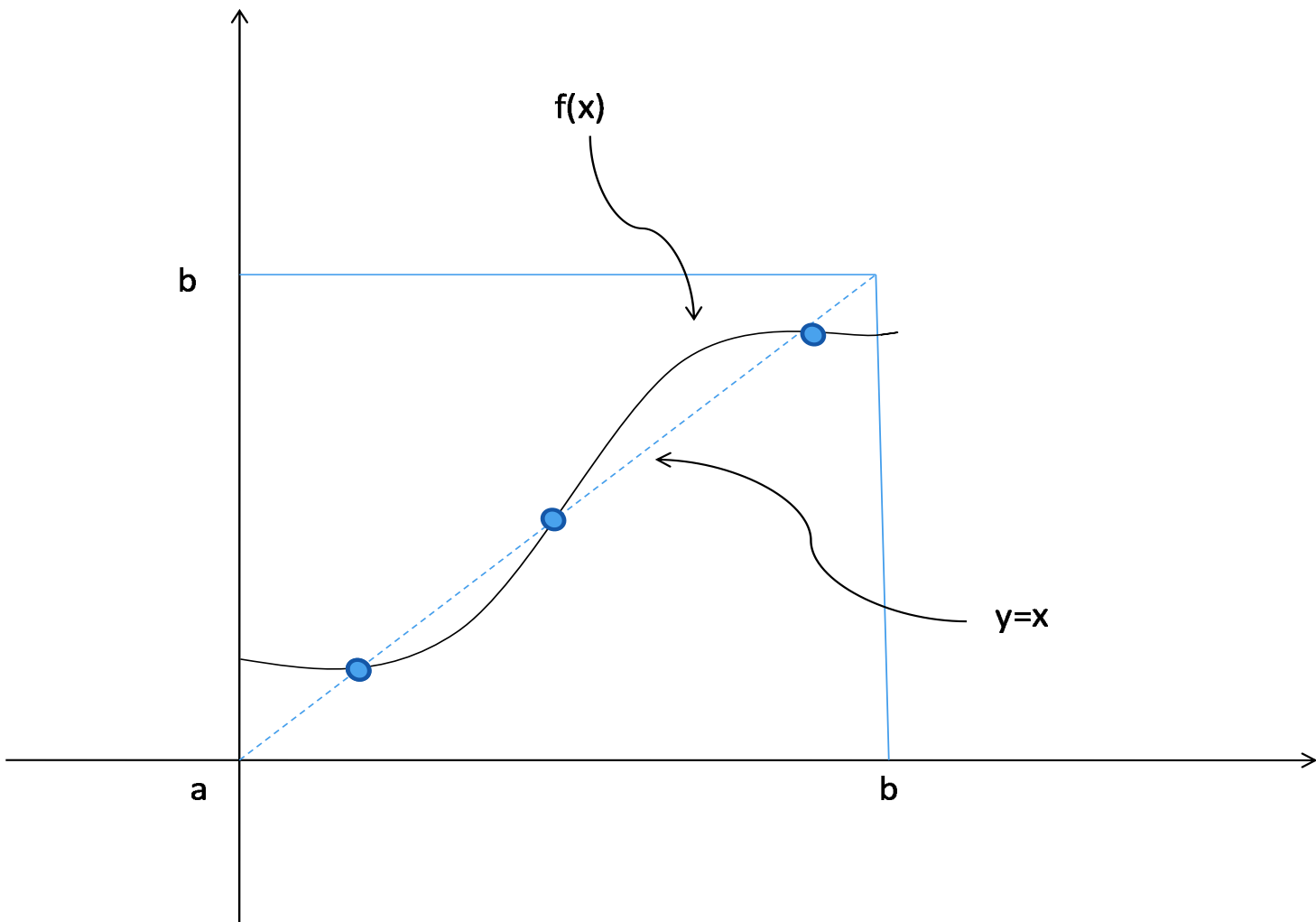
$$g(x) = f(x) - x$$

If either  $f(a) = a$  or  $f(b) = b$ , we're done. So assume  $f(a) > a$  and  $f(b) < b$ . Then

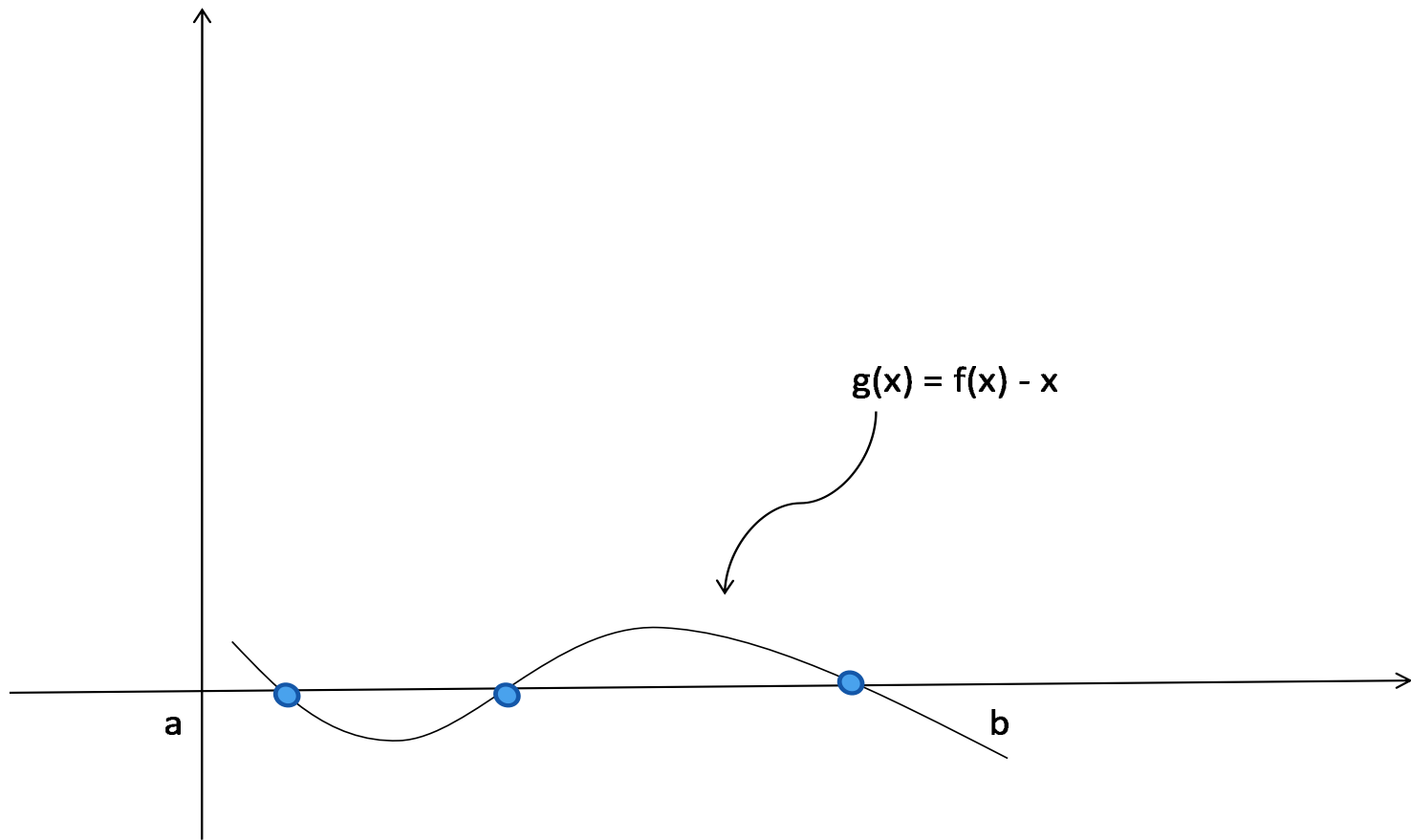
$$g(a) = f(a) - a > 0$$

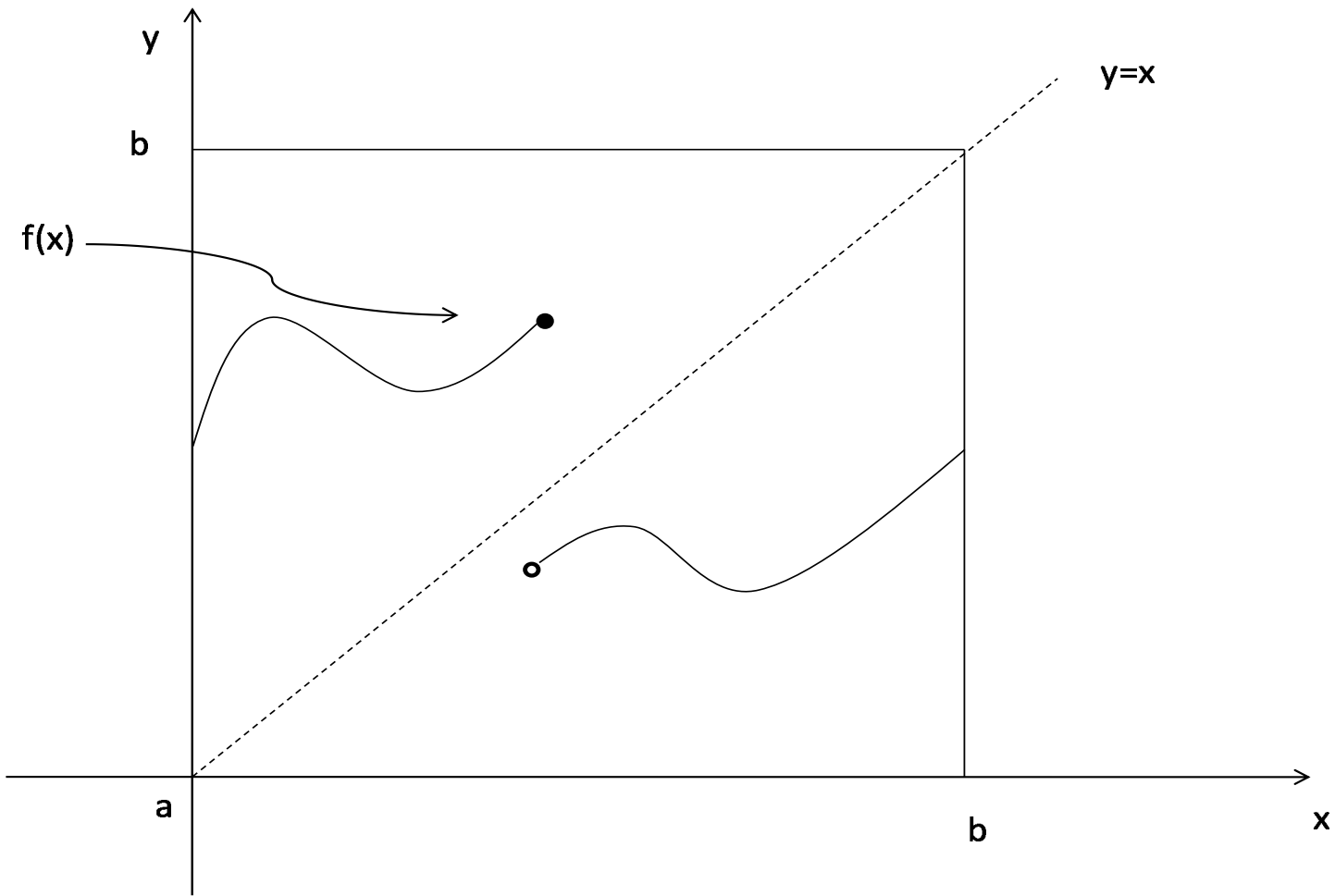
$$g(b) = f(b) - b < 0$$

$g$  is continuous, so by the Intermediate Value Theorem,  $\exists x^* \in (a, b)$  such that  $g(x^*) = 0$ , that is, such that  $f(x^*) = x^*$ .  $\square$









# Brouwer's Fixed Point Theorem

**Theorem 2** (Thm. 3.2. Brouwer's Fixed Point Theorem). *Let  $X \subseteq \mathbf{R}^n$  be nonempty, compact, and convex, and let  $f : X \rightarrow X$  be continuous. Then  $f$  has a fixed point.*

## Sketch of Proof of Brouwer

Consider the case when the set  $X$  is the unit ball in  $\mathbf{R}^n$ , i.e.  $X = B_1[0] = B = \{x \in \mathbf{R}^n : \|x\| \leq 1\}$ . Let  $f : B \rightarrow B$  be a continuous function. Recall that  $\partial B$  denotes the boundary of  $B$ , so  $\partial B = \{x \in \mathbf{R}^n : \|x\| = 1\}$ .

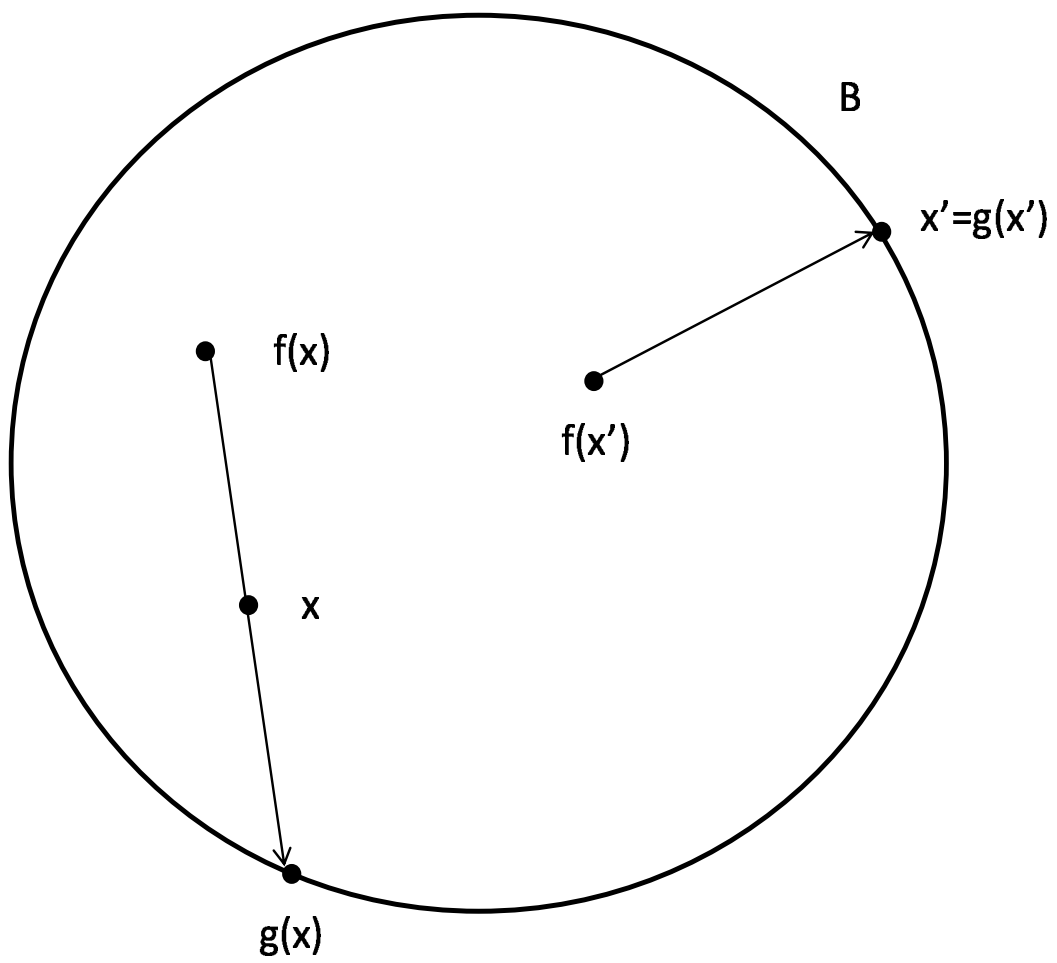
**Fact:** Let  $B$  be the unit ball in  $\mathbf{R}^n$ . Then there is no continuous function  $h : B \rightarrow \partial B$  such that  $h(x') = x'$  for every  $x' \in \partial B$ .

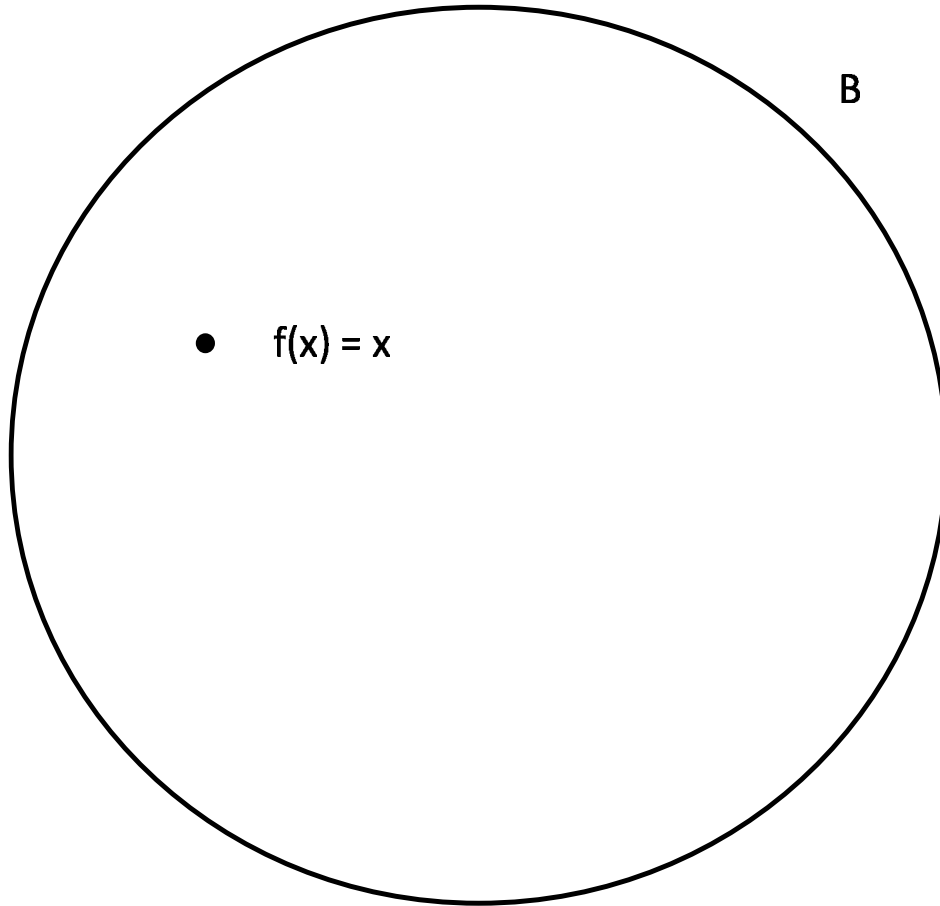
See J. Franklin, *Methods of Mathematical Economics*, for an elementary (but long) proof.

Now to establish Brouwer's theorem, suppose, by way of contradiction, that  $f$  has no fixed points in  $B$ . Thus for every  $x \in B$ ,  $x \neq f(x)$ .

Since  $x \neq f(x)$  for every  $x$ , we can carry out the following construction. For each  $x \in B$ , construct the line segment originating at  $f(x)$  and going through  $x$ . Let  $g(x)$  denote the intersection of this line segment with  $\partial B$ .

This construction is well-defined, and gives a continuous function  $g : B \rightarrow \partial B$ . Furthermore, if  $x' \in \partial B$ , then  $x' = g(x')$ . That is,  $g|_{\partial B} = \text{id}_{\partial B}$ . Since there are no such functions by the fact above, we have a contradiction. Therefore there exists  $x^* \in B$  such that  $f(x^*) = x^*$ , that is,  $f$  has a fixed point in  $B$ .





## Fixed Points for Correspondences

**Definition 2.** Let  $X$  be nonempty and  $\Psi : X \rightarrow 2^X$  be a correspondence. A point  $x^* \in X$  is a fixed point of  $\Psi$  if  $x^* \in \Psi(x^*)$ .

Note here that we do *not* require  $\Psi(x^*) = \{x^*\}$ , that is  $\Psi$  need not be single-valued at  $x^*$ . So  $x^*$  can be a fixed point of  $\Psi$  but there may be other elements of  $\Psi(x^*)$  different from  $x^*$ .



## Examples:

1. Let  $X = [0, 4]$  and  $\Psi : X \rightarrow 2^X$  be given by

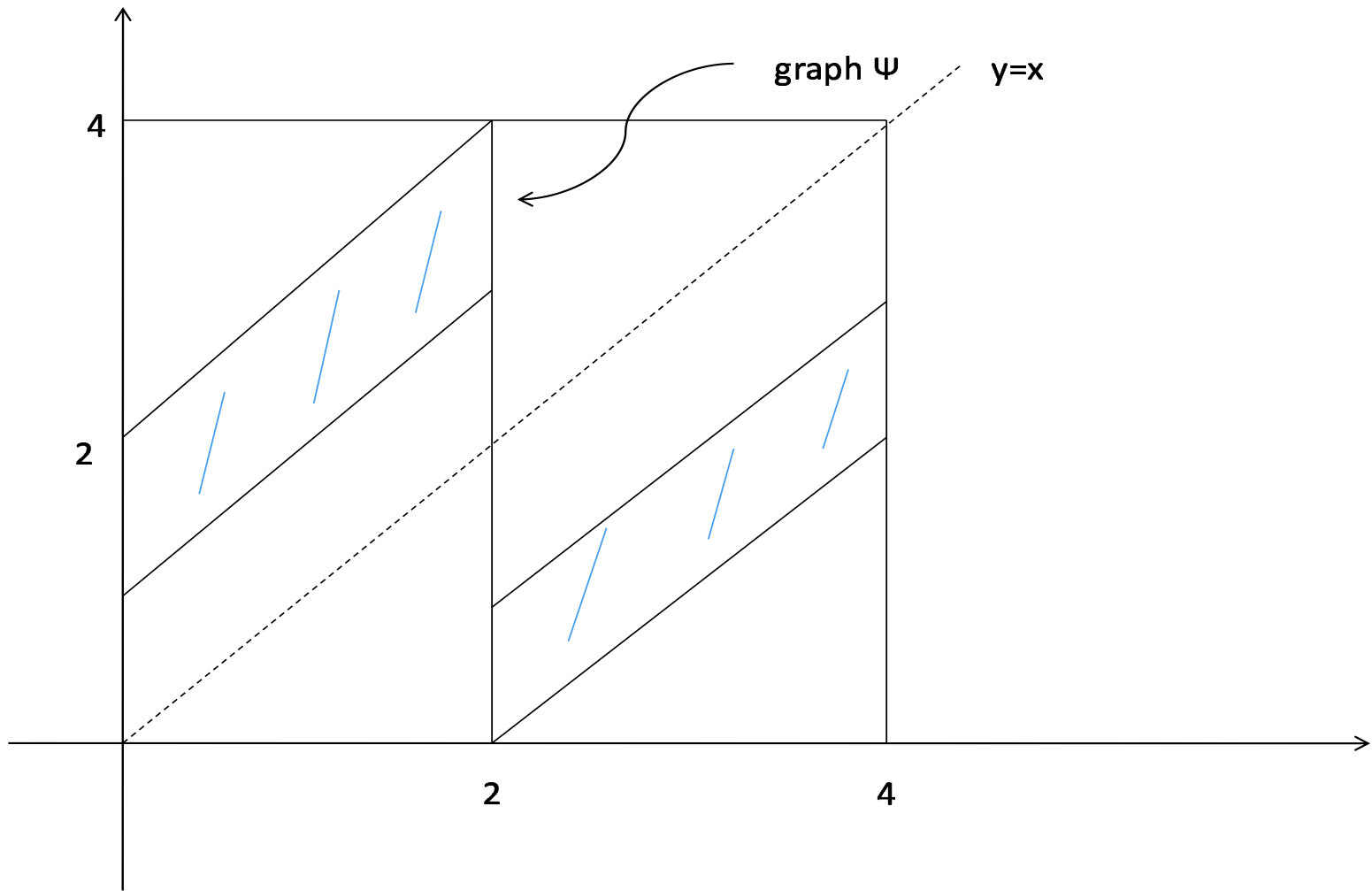
$$\Psi(x) = \begin{cases} [x + 1, x + 2] & \text{if } x < 2 \\ [0, 4] & \text{if } x = 2 \\ [x - 2, x - 1] & \text{if } x > 2 \end{cases}$$

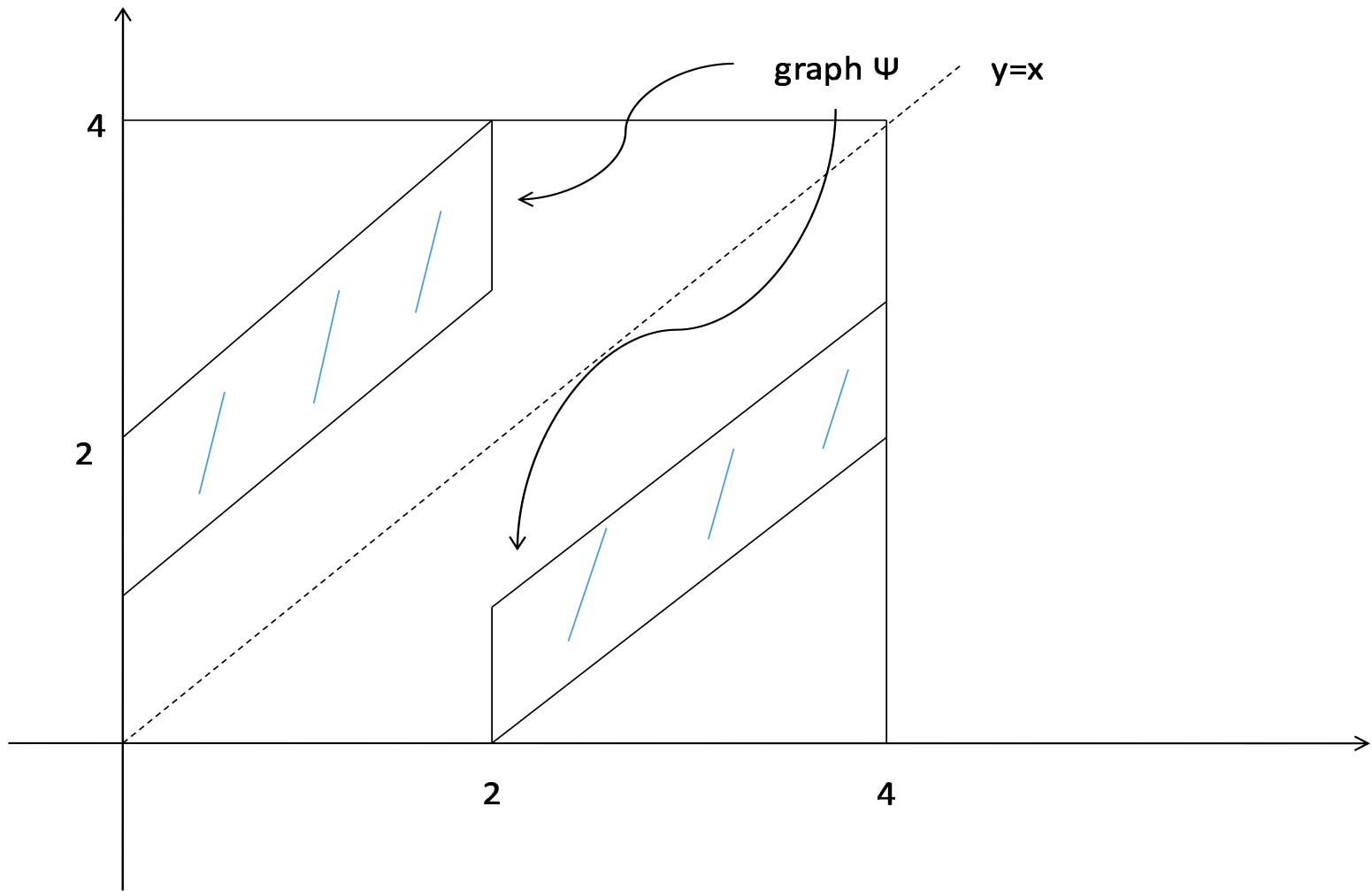
Then  $x = 2$  is the unique fixed point of  $\Psi$ .

2. Let  $X = [0, 4]$  and  $\Psi : X \rightarrow 2^X$  be given by

$$\Psi(x) = \begin{cases} [x + 1, x + 2] & \text{if } x < 2 \\ [0, 1] \cup [3, 4] & \text{if } x = 2 \\ [x - 2, x - 1] & \text{if } x > 2 \end{cases}$$

Then  $\Psi$  has no fixed points.



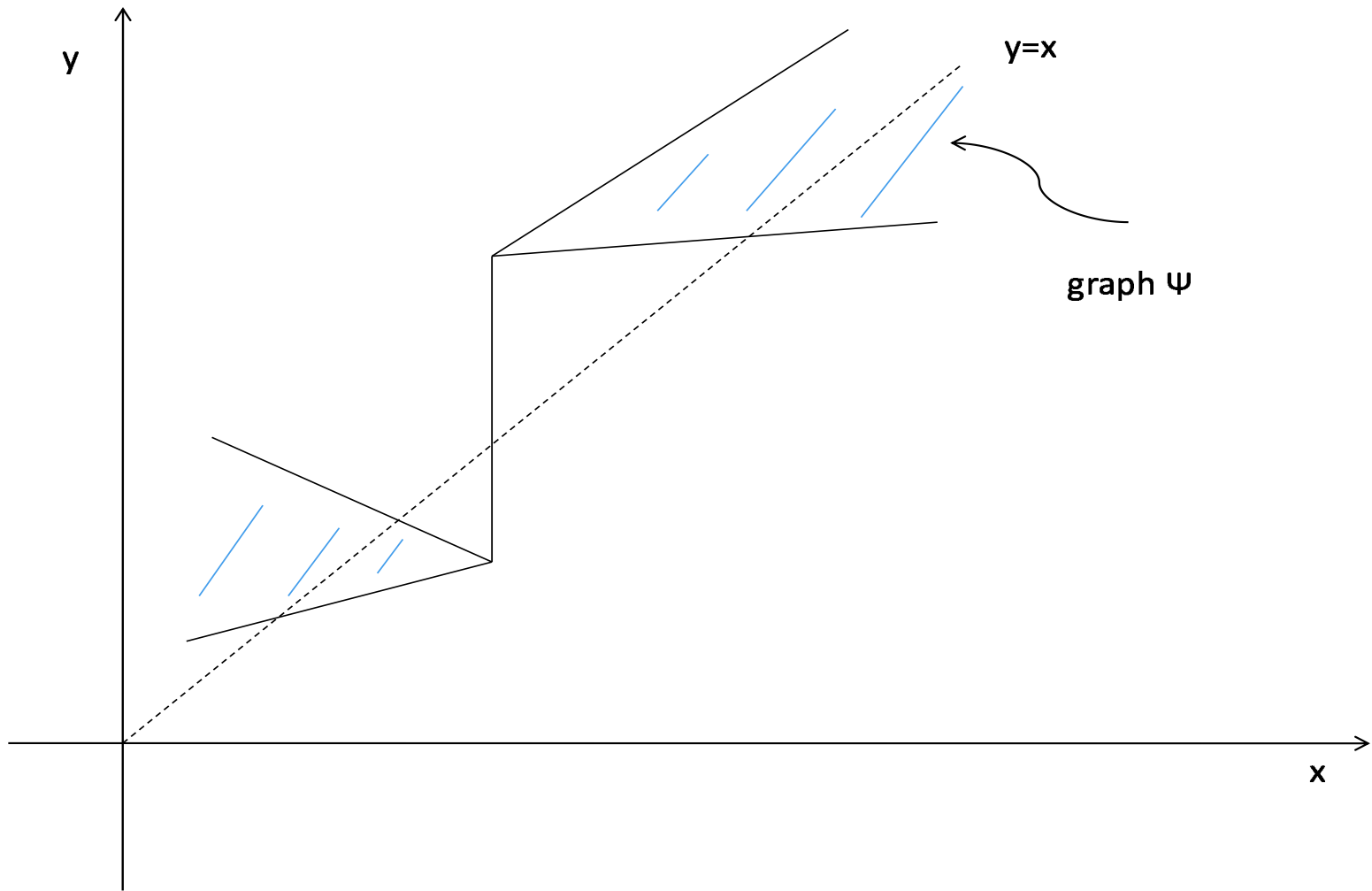


## Kakutani's Fixed Point Theorem

### **Theorem 3. (Thm. 3.4'. Kakutani's Fixed Point Theorem)**

*Let  $X \subseteq \mathbb{R}^n$  be a non-empty, compact, convex set and  $\Psi : X \rightarrow 2^X$  be an upper hemi-continuous correspondence with non-empty, convex, compact values. Then  $\Psi$  has a fixed point in  $X$ .*

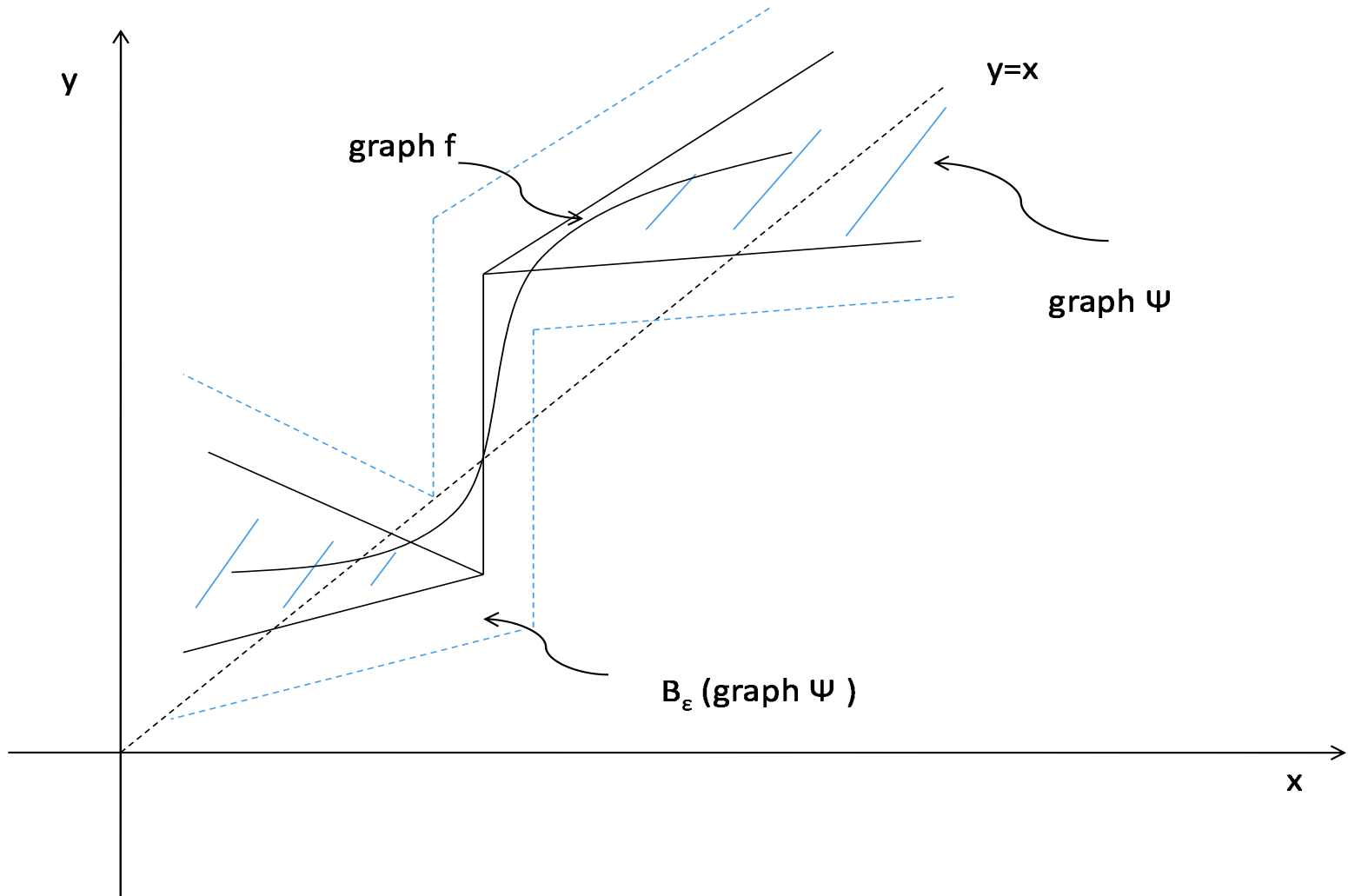
*Proof. (sketch)* Here, the idea is to use Brouwer's theorem after appropriately approximating the correspondence with a function. The catch is that there won't necessarily exist a continuous selection from  $\Psi$ , that is, a continuous function  $f : X \rightarrow X$  such that  $f(x) \in \Psi(x)$  for every  $x \in X$ . If such a function existed, then by applying Brouwer to  $f$  we would have a fixed point of  $\Psi$  (because if  $\exists x^* \in X$  such that  $x^* = f(x^*)$ , then  $x^* = f(x^*) \in \Psi(x^*)$ ).



Instead, we look for a weaker type of approximation. Let  $X \subset \mathbf{R}^n$  be a non-empty, compact, convex set, and let  $\Psi : X \rightarrow 2^X$  be an uhc correspondence with non-empty, compact, convex values. For every  $\varepsilon > 0$ , define the  $\varepsilon$  ball about graph  $\Psi$  to be

$$B_\varepsilon(\text{graph } \Psi) = \left\{ z \in X \times X : d(z, \text{graph } \Psi) = \inf_{(x,y) \in \text{graph } \Psi} d(z, (x,y)) < \varepsilon \right\}$$

Here  $d$  denotes the ordinary Euclidean distance. Since  $\Psi$  is a convex-valued correspondence, for every  $\varepsilon > 0$  there exists a continuous function  $f_\varepsilon : X \rightarrow X$  such that  $\text{graph } f_\varepsilon \subseteq B_\varepsilon(\text{graph } \Psi)$ .



Now by letting  $\varepsilon \rightarrow 0$ , this means that we can find a sequence of continuous functions  $\{f_n\}$  such that  $\text{graph } f_n \subseteq B_{\frac{1}{n}}(\text{graph } \Psi)$  for each  $n$ . By Brouwer's Fixed Point Theorem, each function  $f_n$  has a fixed point  $\hat{x}_n \in X$ , and

$$(\hat{x}_n, \hat{x}_n) = (\hat{x}_n, f_n(\hat{x}_n)) \in \text{graph } f_n \subseteq B_{\frac{1}{n}}(\text{graph } \Psi) \text{ for each } n$$

So for each  $n$  there exists  $(x_n, y_n) \in \text{graph } \Psi$  such that

$$d(\hat{x}_n, x_n) < \frac{1}{n} \text{ and } d(\hat{x}_n, y_n) < \frac{1}{n}$$

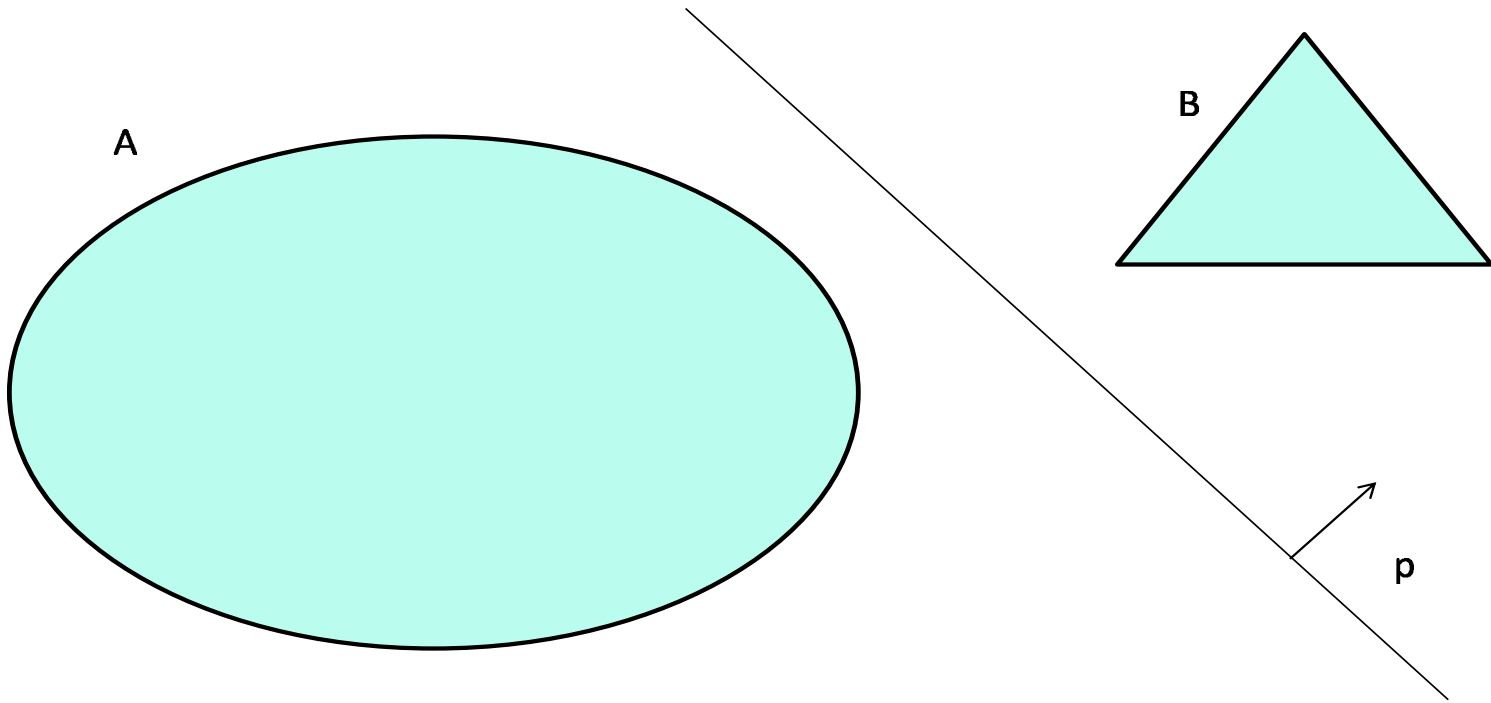
Since  $X$  is compact,  $\{\hat{x}_n\}$  has a convergent subsequence  $\{\hat{x}_{n_k}\}$ , with  $\hat{x}_{n_k} \rightarrow \hat{x} \in X$ . Then  $x_{n_k} \rightarrow \hat{x}$  and  $y_{n_k} \rightarrow \hat{x}$ . Since  $\Psi$  is uhc and closed-valued, it has closed graph, so  $(\hat{x}, \hat{x}) \in \text{graph } \Psi$ . Thus  $\hat{x} \in \Psi(\hat{x})$ , that is,  $\hat{x}$  is a fixed point of  $\Psi$ .  $\square$

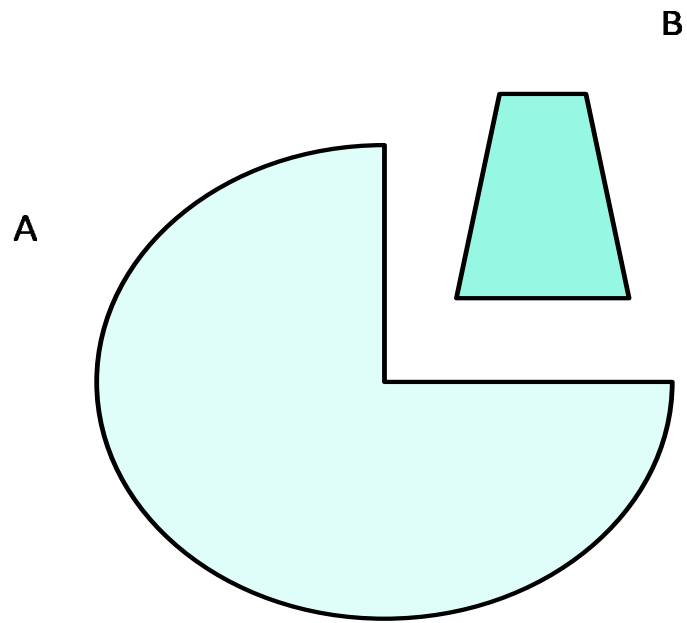


# Separating Hyperplane Theorems

**Theorem 4** (1.26, Separating Hyperplane Theorem). *Let  $A, B \subseteq \mathbf{R}^n$  be nonempty, disjoint convex sets. Then there exists a nonzero vector  $p \in \mathbf{R}^n$  such that*

$$p \cdot a \leq p \cdot b \quad \forall a \in A, b \in B$$





## Separating a Point from a Set

**Theorem 5.** *Let  $Y \subseteq \mathbf{R}^n$  be a nonempty convex set and  $x \notin Y$ . Then there exists a nonzero vector  $p \in \mathbf{R}^n$  such that*

$$p \cdot x \leq p \cdot y \quad \forall y \in Y$$

*Proof.* We sketch the proof in the special case that  $Y$  is compact. We will see that in this case we actually get a stronger conclusion:

$$\exists p \in \mathbf{R}^n, p \neq 0 \text{ s.t. } p \cdot x < p \cdot y \quad \forall y \in Y$$

Choose  $y_0 \in Y$  such that  $|y_0 - x| = \inf\{|y - x| : y \in Y\}$ ; such a point exists because  $Y$  is compact, so the distance function  $g(y) = |y - x|$  assumes its minimum on  $Y$ . Since  $x \notin Y$ ,  $x \neq y_0$ , so  $y_0 - x \neq 0$ . Let  $p = y_0 - x$ . The set

$$H = \{z \in \mathbf{R}^n : p \cdot z = p \cdot y_0\}$$

is the hyperplane perpendicular to  $p$  through  $y_0$ . See Figure 12.  
Then

$$\begin{aligned} p \cdot y_0 &= (y_0 - x) \cdot y_0 \\ &= (y_0 - x) \cdot (y_0 - x + x) \\ &= (y_0 - x) \cdot (y_0 - x) + (y_0 - x) \cdot x \\ &= |y_0 - x|^2 + p \cdot x \\ &> p \cdot x \end{aligned}$$

We claim that

$$y \in Y \Rightarrow p \cdot y \geq p \cdot y_0$$

If not, suppose there exists  $y \in Y$  such that  $p \cdot y < p \cdot y_0$ . Given  $\alpha \in (0, 1)$ , let

$$w_\alpha = \alpha y + (1 - \alpha)y_0$$

Since  $Y$  is convex,  $w_\alpha \in Y$ . Then for  $\alpha$  sufficiently close to zero,

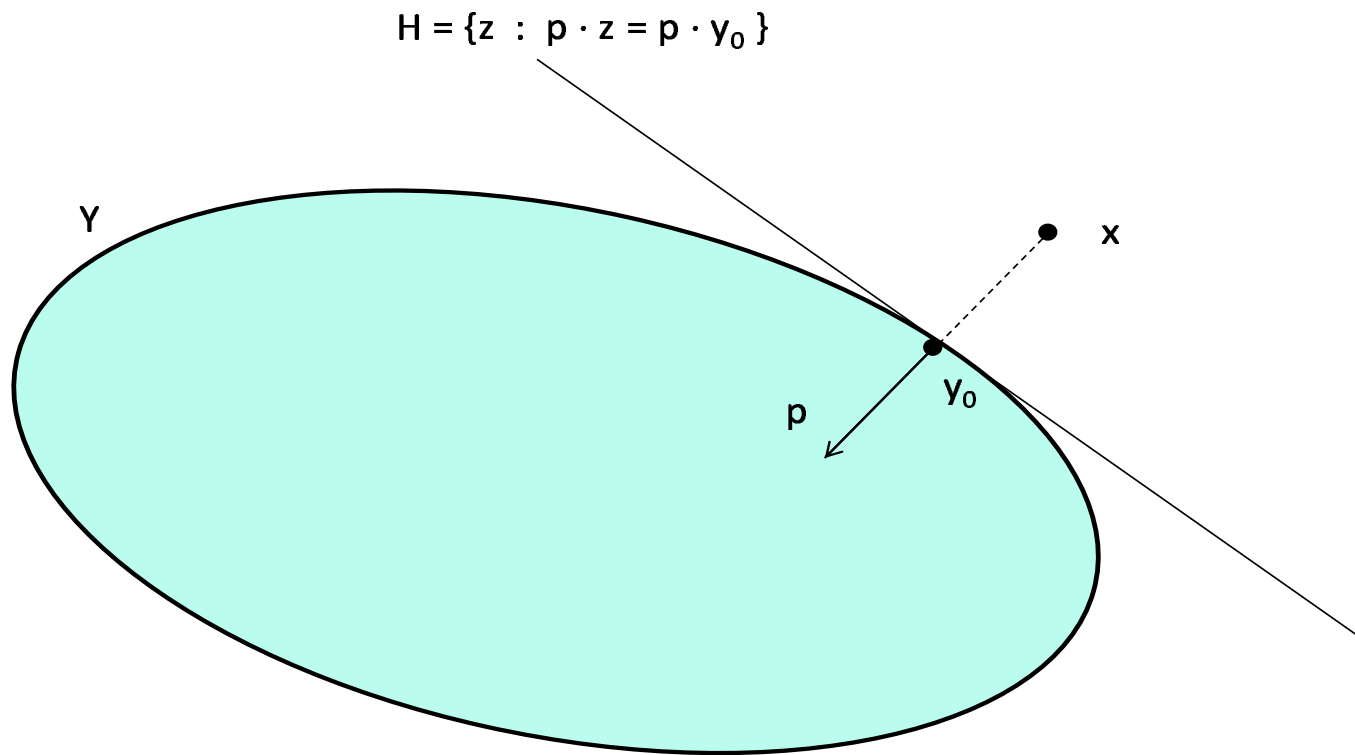
$$\begin{aligned} |x - w_\alpha|^2 &= |x - \alpha y - (1 - \alpha)y_0|^2 \\ &= |x - y_0 + \alpha(y_0 - y)|^2 \\ &= |-p + \alpha(y_0 - y)|^2 \\ &= |p|^2 - 2\alpha p \cdot (y_0 - y) + \alpha^2 |y_0 - y|^2 \\ &= |p|^2 + \alpha (-2p \cdot (y_0 - y) + \alpha |y_0 - y|^2) \\ &< |p|^2 \quad \text{for } \alpha \text{ close to } 0, \text{ as } p \cdot y_0 > p \cdot y \\ &= |y_0 - x|^2 \end{aligned}$$

Thus for  $\alpha$  sufficiently close to zero,

$$|w_\alpha - x| < |y_0 - x|$$

which implies  $y_0$  is not the closest point in  $Y$  to  $x$ , contradiction.

□



The general version of the Separating Hyperplane Theorem can be derived from this special case by noting that if  $A \cap B = \emptyset$ , then  $0 \notin A - B = \{a - b : a \in A, b \in B\}$ .

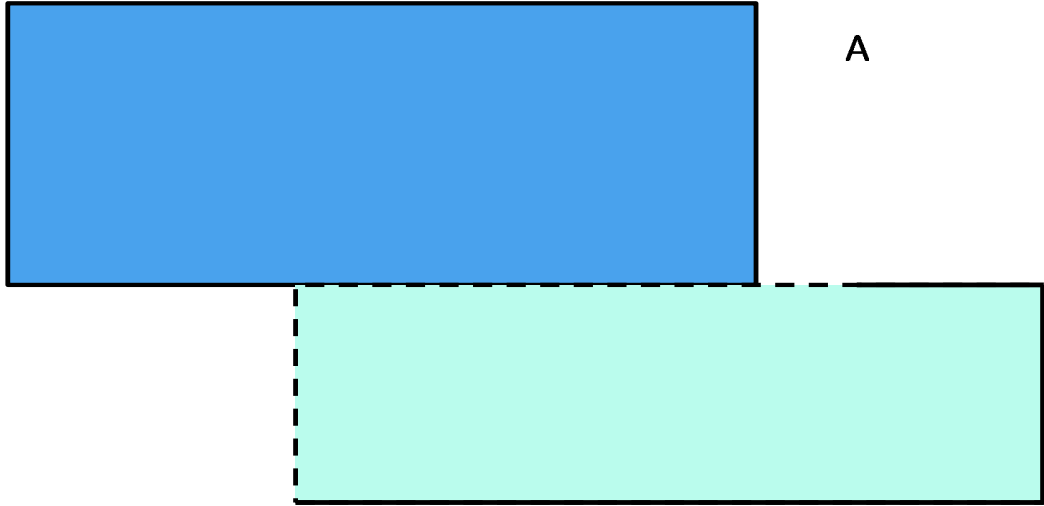


# Strict Separation

For the special case of  $Y$  compact and  $X = \{x\}$ , we actually could *strictly separate*  $Y$  and  $X$ :

$$\exists p \in \mathbf{R}^n, p \neq 0 \text{ s.t. } p \cdot x < p \cdot y \quad \forall y \in Y$$

When can we do this in general? Will require additional assumptions...



A

B

# Strict Separation

**Theorem 6. (Strict Separating Hyperplane Theorem)** *Let  $A, B \subseteq \mathbf{R}^n$  be nonempty, disjoint, closed, convex sets. Then there exists a nonzero vector  $p \in \mathbf{R}^n$  such that*

$$p \cdot a < p \cdot b \quad \forall a \in A, b \in B$$