1. Introductions
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Introductions

Welcome

• 204
• Berkeley Economics
• UC Berkeley
• California
• US...
Introductions

• Chris Shannon

• Farzad Pourbabaee

• Damian Vergara
About the Course

- **Schedule:** Lectures MTWThF 9:00 - 11:30 am (Berkeley time), often going over so don’t schedule anything before 12:00. Videos posted in courses folder “lecture videos”.

- **Discussion Sections:** MTWThF 1:00 - 3:00 pm (first section today).

- **Office hours:** Chris Shannon MTWThF 11:30 - 12:30 (end of lecture + 1 hour), also by appointment.

- **Final Exam:** Wednesday August 19, open book + notes, 24 hour.
• **Prerequisites:** Math 53-54 at Berkeley or equivalent
  
  – 4 semesters college mathematics
  
  – linear algebra
  
  – multivariable calculus
  
  – rigorous approach - theorems stated carefully and some proofs given
  
  – stream for engineers and scientists
Course requirements:

• problems sets: 6 total
  
  (no late problem sets...no exceptions)

• exam

• reading/working on your own

Grade: 10% problem sets (5 highest scores out of 6), 90% final exam
Grading in First Year Economics Courses:

- median grade = B+: solid command of material

- A and A- are very good grades, A+ for truly exceptional work

- B: ready to go on to further work...a B in 204 means you are ready to go on to 201a/b, 202a/b, 240a/b

- B-: very marginal, but we won’t make you take the class again. B- in 204 means you will have a very hard time in 201a/b. Recommend you take Math 53 and 54 this year, maybe Math 104, come back next year to retake 204 and
take 201a/b. B- is a passing grade, but you must maintain a B average

- C: not passing. Definitely not ready for 201a/b, 202a/b, 240a/b. Take Math 53-54 this year, maybe Math 104, retake 204 next year

- 204 with at least a B- (or a waiver from 204 requirement) is a strictly enforced prerequisite for enrollment in 201a/b

- F: means you didn’t take the final exam. Be sure to withdraw if you don’t or can’t take the final.
This year we strongly recommend all students take 204 S/U (pass/no pass)
Resources:

Book: de la Fuente, *Mathematical Methods and Models for Economists*

Lecture notes: for every lecture + supplements for several topics

*Be sure to read Corrections Handout with dlF*

Seek out other references
This class is not normal.

- lectures
- expectations
Goals for 204

- reduce heterogeneity of math backgrounds for students in Econ graduate classes

- advance everyone’s math skills and knowledge

- present some particular concepts and results used in first-year economics courses 201a/b, 202a/b

- challenge everyone - so not everyone will understand everything
• develop basic math skills and knowledge needed to work as a professional economist and read academic economics

• develop ability to read and evaluate purported proofs...essential for reading and working in all branches of economics - theoretical, empirical, experimental

• develop ability to compose simple proofs...essential to working in all branches of economics - theoretical, empirical, experimental

• cover selected material from real analysis and linear algebra at moderate level of abstraction (considerably more advanced and abstract than Math 53 + 54)
• **not** to review Math 53 + 54. If you are weak on this material, take Math 53-54 this year, and take 204 next year.
Learning by Doing

- to learn this sort of mathematics you need to do more than just read the book and notes and listen to lectures
- active reading: work through each line, be sure you know how to get from one line to the next
- active listening: follow each step as we work through arguments in class
- working problems: the most valuable part of the class
• working in groups strongly encouraged...

• but, always try to work through all of the problems before talking to others

• everyone must write up his/her own solutions

• best test of understanding: can you explain it to others
Methods of Proof

What is a proof? The million dollar question...

Main Methods of Proof:

• deduction

• contraposition

• induction

• contradiction
We’ll examine each of these in turn.
Proof by Deduction: A list of statements, the last of which is the statement to be proven. Each statement in the list is either

- an axiom: a fundamental assumption about mathematics, or part of definition of the object under study; or

- a previously established theorem; or

- follows from previous statements in the list by a valid rule of inference
Proof by Deduction

Example: Prove that the function \( f(x) = x^2 \) is continuous at \( x = 5 \).

Recall from one-variable calculus that \( f(x) = x^2 \) is continuous at \( x = 5 \) means

\[
\forall \varepsilon > 0 \exists \delta > 0 \text{ s.t. } |x - 5| < \delta \Rightarrow |f(x) - f(5)| < \varepsilon
\]

That is, “for every \( \varepsilon > 0 \) there exists a \( \delta > 0 \) such that whenever \( x \) is within \( \delta \) of 5, \( f(x) \) is within \( \varepsilon \) of \( f(5) \).”

To prove the claim, we must systematically verify that this definition is satisfied.
Proof. Let $\varepsilon > 0$ be given. Let

$$\delta = \min \left\{1, \frac{\varepsilon}{11} \right\} > 0 \quad \Rightarrow \quad \delta \leq \frac{\varepsilon}{11}$$

Where did that come from? Suppose $|x - 5| < \delta$. Since $\delta \leq 1$, $4 < x < 6$, so $9 < x + 5 < 11$ and $|x + 5| < 11$. Then

$$|f(x) - f(5)| = |x^2 - 25| = |(x + 5)(x - 5)| = |x + 5||x - 5| \leq 11 \cdot \delta \leq 11 \cdot \frac{\varepsilon}{11} = \varepsilon$$

Thus, we have shown that for every $\varepsilon > 0$, there exists $\delta > 0$ such that $|x - 5| < \delta \Rightarrow |f(x) - f(5)| < \varepsilon$, so $f$ is continuous at $x = 5$. \qed
Proof by Contraposition

Recall some basics of logic.

\( \neg P \) means “\( P \) is false.”

\( P \land Q \) means “\( P \) is true and \( Q \) is true.”

\( P \lor Q \) means “\( P \) is true or \( Q \) is true (or possibly both).”

\( \neg P \land Q \) means \( (\neg P) \land Q \); \( \neg P \lor Q \) means \( (\neg P) \lor Q \).

\( P \Rightarrow Q \) means “whenever \( P \) is satisfied, \( Q \) is also satisfied.”

Formally, \( P \Rightarrow Q \) is equivalent to \( \neg P \lor Q \).
Proof by Contraposition

The contrapositive of the statement \( P \Rightarrow Q \) is the statement \( \neg Q \Rightarrow \neg P \).

**Theorem 1.** \( P \Rightarrow Q \) is true if and only if \( \neg Q \Rightarrow \neg P \) is true.

**Proof.** Suppose \( P \Rightarrow Q \) is true. Then either \( P \) is false, or \( Q \) is true (or possibly both). Therefore, either \( \neg P \) is true, or \( \neg Q \) is false (or possibly both), so \( \neg (\neg Q) \vee (\neg P) \) is true, that is, \( \neg Q \Rightarrow \neg P \) is true.

Conversely, suppose \( \neg Q \Rightarrow \neg P \) is true. Then either \( \neg Q \) is false, or \( \neg P \) is true (or possibly both), so either \( Q \) is true, or \( P \) is false (or possibly both), so \( \neg P \vee Q \) is true, so \( P \Rightarrow Q \) is true. \( \square \)
Proof by Induction

We illustrate with an example:

**Theorem 2.** For every \( n \in \mathbb{N}_0 = \{0, 1, 2, 3, \ldots \} \),

\[
\sum_{k=1}^{n} k = \frac{n(n+1)}{2}
\]

i.e. \( 1 + 2 + \cdots + n = \frac{n(n+1)}{2} \).

*Proof. Base step* \( n = 0 \): LHS = \( \sum_{k=1}^{0} k \) = the empty sum = 0. RHS = \( \frac{0 \cdot 1}{2} = 0 \)

So the claim is true for \( n = 0 \).
**Induction step:** Suppose

\[ \sum_{k=1}^{n} k = \frac{n(n + 1)}{2} \]

for some \( n \geq 0 \)

We must show that

\[ \sum_{k=1}^{n+1} k = \frac{(n + 1)((n + 1) + 1)}{2} \]
LHS \quad = \quad \sum_{k=1}^{n+1} \frac{n+1}{2} + (n + 1) \\
\quad = \quad \frac{n(n + 1)}{2} + (n + 1) \text{ by the Induction hypothesis} \\
\quad = \quad (n + 1)\left(\frac{n}{2} + 1\right) \\
\quad = \quad \frac{(n + 1)(n + 2)}{2} \\
RHS \quad = \quad \frac{(n + 1)((n + 1) + 1)}{2} \\
\quad = \quad \frac{(n + 1)(n + 2)}{2} = \text{LHS} \\
So by mathematical induction, \sum_{k=1}^{n} k = \frac{n(n+1)}{2} \text{ for all } n \in \mathbb{N}_0.
Proof by Contradiction

Assume the negation of what is claimed, and work toward a contradiction.

**Theorem 3.** There is no rational number $q$ such that $q^2 = 2$.

*Proof.* Suppose $q^2 = 2$ where $q \in \mathbb{Q}$. Then we can write $q = \frac{m}{n}$ for some integers $m, n \in \mathbb{Z}$. Moreover, we can assume that $m$ and $n$ have no common factor; if they did, we could divide it out.

$$2 = q^2 = \frac{m^2}{n^2}$$

Therefore, $m^2 = 2n^2$, so $m^2$ is even.
We claim that $m$ is even. If not, then $m$ is odd, so $m = 2p + 1$ for some $p \in \mathbb{Z}$. Then

$$m^2 = (2p + 1)^2 = 4p^2 + 4p + 1 = 2(2p^2 + 2p) + 1$$

which is odd, contradiction. Therefore, $m$ is even, so $m = 2r$ for some $r \in \mathbb{Z}$.

$$m^2 = 4r^2 = (2r)^2 = m^2 = 2n^2$$

So $n^2$ is even, which implies (by the argument given above) that $n$ is even. Therefore, $n = 2s$ for some $s \in \mathbb{Z}$, so $m$ and $n$ have a
common factor, namely 2, contradiction. Therefore, there is no rational number $q$ such that $q^2 = 2$. \qed
Equivalence Relations

Definition 1. A binary relation $R$ from $X$ to $Y$ is a subset $R \subseteq X \times Y$. We write $xRy$ if $(x, y) \in R$ and “not $xRy$” if $(x, y) \notin R$. $R \subseteq X \times X$ is a binary relation on $X$.

Example: Suppose $f : X \to Y$ is a function from $X$ to $Y$. The binary relation $R \subseteq X \times Y$ defined by

$$xRy \iff f(x) = y$$

is exactly the graph of the function $f$. A function can be considered a binary relation $R$ from $X$ to $Y$ such that for each $x \in X$ there exists exactly one $y \in Y$ such that $(x, y) \in R$.

Example: Suppose $X = \{1, 2, 3\}$ and $R$ is the binary relation on $X$ given by $R = \{(1, 1), (2, 1), (2, 2), (3, 1), (3, 2), (3, 3)\}$. This is the binary relation “is weakly greater than,” or $\geq$. 
Equivalence Relations

**Definition 2.** A binary relation $R$ on $X$ is

(i) reflexive if $\forall x \in X, xRx$

(ii) symmetric if $\forall x, y \in X, xRy \iff yRx$

(iii) transitive if $\forall x, y, z \in X, (xRy \land yRz) \Rightarrow xRz$

**Definition 3.** A binary relation $R$ on $X$ is an equivalence relation if it is reflexive, symmetric and transitive.
Equivalence Relations

Definition 4. Given an equivalence relation \( R \) on \( X \), write
\[
[x] = \{ y \in X : xRy \}
\]

\([x]\) is called the equivalence class containing \( x \).

The set of equivalence classes is the quotient of \( X \) with respect to \( R \), denoted \( X/R \).

Example: The binary relation \( \geq \) on \( \mathbb{R} \) is not an equivalence relation because it is not symmetric.

Example: Let \( X = \{a, b, c, d\} \) and
\[
R = \{(a, a), (a, b), (b, a), (b, b), (c, c), (c, d), (d, c), (d, d)\}
\]
\( R \) is an equivalence relation (why?) and the equivalence classes of \( R \) are \( \{a, b\} \) and \( \{c, d\} \). \( X/R = \{\{a, b\}, \{c, d\}\} \).
$R = X \times X$

$(a, a), (c, a), (a, c)$

$\{a\} = \{a, b, c, d\}$

$X / R = \{\{a, b, c, d\}\}$
Equivalence Relations

The equivalence classes of an equivalence relation form a \textit{partition} of $X$: every element of $X$ belongs to exactly one equivalence class.

\textbf{Theorem 4.} Let $R$ be an equivalence relation on $X$. Then $\forall x \in X, x \in [x]$. Given $x, y \in X$, either $[x] = [y]$ or $[x] \cap [y] = \emptyset$.

\textit{Proof.} If $x \in X$, then $xRx$ because $R$ is reflexive, so $x \in [x]$.

Suppose $x, y \in X$. If $[x] \cap [y] = \emptyset$, we’re done. So suppose $[x] \cap [y] \neq \emptyset$. We must show that $[x] = [y]$, i.e. that the elements of $[x]$ are exactly the same as the elements of $[y]$. 

Choose \( z \in [x] \cap [y] \). Then \( z \in [x] \), so \( xRz \). By symmetry, \( zRx \). Also \( z \in [y] \), so \( yRz \). By symmetry again, \( zRy \). Now choose \( w \in [x] \). By definition, \( xRw \). Since \( zRx \) and \( R \) is transitive, \( zRw \). By symmetry, \( wRz \). Since \( zRy \), \( wRy \) by transitivity again. By symmetry, \( yRw \), so \( w \in [y] \), which shows that \( [x] \subseteq [y] \).

Similarly, \( [y] \subseteq [x] \), so \( [x] = [y] \). \( \square \)
Cardinality

**Definition 5.** Two sets $A, B$ are numerically equivalent (or have the same cardinality) if there is a bijection $f : A \to B$, that is, a function $f : A \to B$ that is 1-1 ($a \neq a' \Rightarrow f(a) \neq f(a')$), and onto ($\forall b \in B \exists a \in A \text{ s.t. } f(a) = b$).

**Example:** $A = \{2, 4, 6, \ldots, 50\}$ is numerically equivalent to the set $\{1, 2, \ldots, 25\}$ under the function $f(n) = 2n$.

$B = \{1, 4, 9, 16, 25, 36, 49 \ldots\} = \{n^2 : n \in \mathbb{N}\}$ is numerically equivalent to $\mathbb{N}$. 
Cardinality

A set is either finite or infinite. A set is finite if it is numerically equivalent to \( \{1, \ldots, n\} \) for some \( n \). A set that is not finite is infinite.

In particular, \( A = \{2, 4, 6, \ldots, 50\} \) is finite, \( B = \{1, 4, 9, 16, 25, 36, 49 \ldots\} \) is infinite.

A set is countable if it is numerically equivalent to the set of natural numbers \( \mathbb{N} = \{1, 2, 3, \ldots\} \). An infinite set that is not countable is called uncountable.
Cardinality

**Example:** The set of integers $\mathbb{Z}$ is countable.

$$\mathbb{Z} = \{0, 1, -1, 2, -2, \ldots\}$$

Define $f : \mathbb{N} \to \mathbb{Z}$ by

$$
\begin{align*}
    f(1) & = 0 \\
    f(2) & = 1 \\
    f(3) & = -1 \\
    \vdots \\
    f(n) & = (-1)^n \left\lfloor \frac{n}{2} \right\rfloor
\end{align*}
$$

where $[x]$ is the greatest integer less than or equal to $x$. It is straightforward to verify that $f$ is one-to-one and onto.
Cardinality

**Theorem 5.** The set of rational numbers $\mathbb{Q}$ is countable.

“Picture Proof”:

\[
\mathbb{Q} = \left\{ \frac{m}{n} : m, n \in \mathbb{Z}, n \neq 0 \right\} = \left\{ \frac{m}{n} : m \in \mathbb{Z}, n \in \mathbb{N} \right\}
\]
Go back and forth on upward-sloping diagonals, omitting the
$f : \mathbb{N} \to \mathbb{Q}$, \( f \) is one-to-one and onto.

<table>
<thead>
<tr>
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<td>1</td>
<td>1</td>
<td>$\frac{-1}{2}$</td>
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