Econ 204 2020

Lecture 7

Outline

1. Connected Sets
2. Correspondences
3. Continuity for Correspondences

Announcements

- PS 2 due today 1 pm
  - $\rightsquigarrow$ solns posted $\approx$ 2 pm
- PS 3 due Friday 8/7 1 pm

Connected Sets

Definition 1. *Two sets* A, B *in a metric space are* separated *if*

$$\bar{A} \cap B = A \cap \bar{B} = \emptyset$$

A set in a metric space is connected *if it cannot be written as the union of two nonempty separated sets.*

Connected Sets

in $\mathbb{R}$ with standard metric

Example: $[0,1)$ and $[1,2]$ are disjoint but not separated:

$$\overline{[0,1)} \cap [1,2] = [0,1] \cap [1,2] = \{1\} \neq \emptyset$$

$[0,1)$ and $(1,2]$ are separated:

$$\begin{aligned} \overline{[0,1)} \cap (1,2] &= [0,1] \cap (1,2] = \emptyset \\ [0,1) \cap \overline{(1,2]} &= [0,1) \cap [1,2] = \emptyset \end{aligned}$$

Note that $d([0,1),(1,2]) = 0$ even though the sets are separated.

Connected Sets

- Note that separation does *not* require that $\bar{A} \cap \bar{B} = \emptyset$. For example,

$$A = [0,1) \cup (1,2]$$

  is not connected.

- A common equivalent definition: A set Y in a metric space X is connected if there do not exist open sets A and B such that $A \cap B = \emptyset$, $Y \subseteq A \cup B$ and $A \cap Y \neq \emptyset$ and $B \cap Y \neq \emptyset$.

Connected Sets

$\mathbb{R}$ with standard metric

Theorem 1 (Thm. 9.2). *A set $S \subseteq \mathbf{E}^1$ is connected if and only if it is an interval, i.e. if $x, y \in S$ and $z \in (x, y)$, then $z \in S$.*

$\Rightarrow$: *Proof.* First, we show that if S is connected then S is an interval. We do this by proving the contrapositive: if S is not an interval, then it is not connected. If S is not an interval, find

$$x, y \in S,\ x < z < y,\ z \notin S$$

Let

$$A = S \cap (-\infty, z),\ B = S \cap (z, \infty)$$

Then

$$\begin{aligned}
\bar{A} \cap B &\subseteq \overline{(-\infty, z)} \cap (z, \infty) = (-\infty, z] \cap (z, \infty) = \emptyset \\
A \cap \bar{B} &\subseteq (-\infty, z) \cap \overline{(z, \infty)} = (-\infty, z) \cap [z, \infty) = \emptyset \\
A \cup B &= (S \cap (-\infty, z)) \cup (S \cap (z, \infty)) \\
&= S \setminus \{z\} \\
&= S \\
x &\in A, \text{ so } A \neq \emptyset \\
y &\in B, \text{ so } B \neq \emptyset
\end{aligned}$$

So S is not connected. We have shown that if S is not an interval, then S is not connected; therefore, if S is connected, then S is an interval.

$\Leftarrow$: Now, we need to show that if S is an interval, it is connected. This is much like the proof of the Intermediate Value Theorem. See de la Fuente for the details. $\square$

Continuity and Connectedness

In a general metric space, continuity will preserve connectedness.

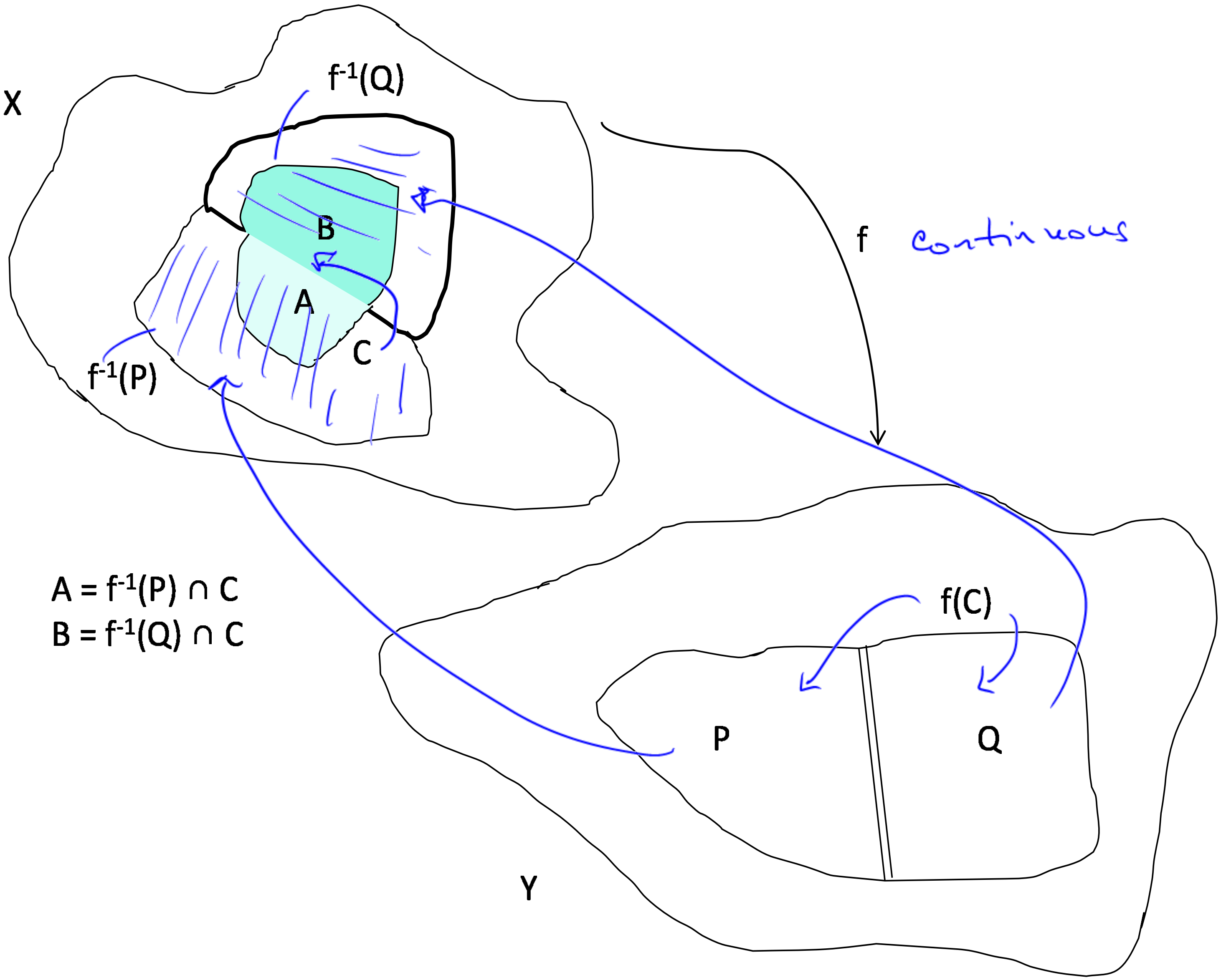
Theorem 2 (Thm. 9.3)**.** *Let X and Y be metric spaces and $f : X \to Y$ be continuous. If C is a connected subset of X, then $f(C)$ is* a *connected.* subset of Y.

Proof. We prove the contrapositive: if $f(C)$ is not connected, then C is not connected. Suppose $f(C)$ is not connected. Then there exist P, Q such that $P \neq \emptyset \neq Q$, $f(C) = P \cup Q$, and

$$\bar{P} \cap Q = P \cap \bar{Q} = \emptyset$$

Let

$$A = f^{-1}(P) \cap C \text{ and } B = f^{-1}(Q) \cap C$$



$A = f^{-1}(P) \cap C$
$B = f^{-1}(Q) \cap C$

$S \subseteq T$, T closed

Then $\Rightarrow \bar{S} \subseteq T$

$$\begin{aligned} A \cup B &= \left(f^{-1}(P) \cap C\right) \cup \left(f^{-1}(Q) \cap C\right) \\ &= \left(f^{-1}(P) \cup f^{-1}(Q)\right) \cap C \\ &= f^{-1}(P \cup Q) \cap C \\ &= f^{-1}(f(C)) \cap C \qquad C \subseteq f^{-1}(f(C)) \\ &= C \end{aligned}$$

Also, $A = f^{-1}(P) \cap C \neq \emptyset$ and $B = f^{-1}(Q) \cap C \neq \emptyset$. Then note

$$A = f^{-1}(P) \cap C \subseteq f^{-1}(P) \subseteq f^{-1}\left(\bar{P}\right)$$

Since f is continuous, $f^{-1}(\bar{P})$ is closed, so

$$\bar{A} \subseteq f^{-1}\left(\bar{P}\right)$$

Similarly,

$$B = f^{-1}(Q) \cap C \subseteq f^{-1}(Q) \subseteq f^{-1}\left(\bar{Q}\right)$$

and $f^{-1}(\bar{Q})$ is closed, so

$$\bar{B} \subseteq f^{-1}(\bar{Q})$$

Then

$$\begin{aligned} \bar{A} \cap B &\subseteq f^{-1}(\bar{P}) \cap f^{-1}(Q) \\ &= f^{-1}(\bar{P} \cap Q) \\ &= f^{-1}(\emptyset) \\ &= \emptyset \end{aligned}$$

and similarly

$$\begin{aligned} A \cap \bar{B} &\subseteq f^{-1}(P) \cap f^{-1}(\bar{Q}) \\ &= f^{-1}(P \cap \bar{Q}) \\ &= f^{-1}(\emptyset) \\ &= \emptyset \end{aligned}$$

So C is not connected. We have shown that $f(C)$ not connected implies C not connected; therefore, C connected implies $f(C)$ connected. $\square$

You can view this result as a generalization of the Intermediate Value Theorem.

Intermediate Value Theorem, Yet Again

This lets us give a third, and slickest, proof of the Intermediate Value Theorem.

$\subseteq \mathbb{R}$

Corollary 1 (Intermediate Value Theorem)**.** *If* $f : [a, b] \to \mathbf{R}$ *is continuous, and* $f(a) < d < f(b)$*, then there exists* $c \in (a, b)$ *such that* $f(c) = d$*.*

Proof. Since $[a, b]$ is an interval in $\mathbb{R}$, it is connected. So $f([a, b])$ is connected, hence $f([a, b])$ is an interval. $f(a) \in f([a, b])$, and $f(b) \in f([a, b])$, and $d \in [f(a), f(b)]$; since $f([a, b])$ is an interval, $d \in f([a, b])$, i.e. there exists $c \in [a, b]$ such that $f(c) = d$. Since $f(a) < d < f(b)$, $c \neq a$, $c \neq b$, so $c \in (a, b)$. $\square$

Recall: 2^A - set of all subsets of A

Correspondences

Definition 2. *A* correspondence $\Psi : X \to 2^Y$ *from* X *to* Y *is a function from* X *to* 2^Y*, that is,* $\Psi(x) \subseteq Y$ *for every* $x \in X$*.*

other common notation:

$$\Psi : X \rightrightarrows Y$$

$$\Psi : X \to Y \quad \text{a correspondence}$$

$$\Psi : X \rightsquigarrow Y$$

Correspondences

Examples:

1. Let $u : \mathbf{R}^n_+ \to \mathbf{R}$ be a continuous utility function, $y > 0$ and $p \in \mathbf{R}^n_{++}$, that is, $p_i > 0$ for each i.

   Define $\Psi : \mathbf{R}^n_{++} \times \mathbf{R}_{++} \to 2^{\mathbf{R}^n_+}$ by

   $$\Psi(p, y) = \operatorname{arg\,max} u(x) \quad \text{s.t. } x \geq 0, \quad \sum_{i=1}^{n} p_i x_i = p \cdot x \leq y$$

   "argument(s) that maximize u over constraint set"

   Ψ is the *demand correspondence* associated with the utility function u; typically $\Psi(p, y)$ is multi-valued.

That is, with

$$B(p,y) = \{x \in \mathbb{R}^n : x \geq 0,\ p \cdot x \leq y\}$$

$$\Psi(p,y) = \{x^* \in B(p,y) : u(x^*) \geq u(x)\ \forall x \in B(p,y)\}$$

2. Let $f : X \to Y$ be a function. Define $\Psi : X \to 2^Y$ by

$$\Psi(x) = \{f(x)\} \text{ for each } x \in X$$

That is, we can consider a function to be the special case of a correspondence that is single-valued for each x.

Continuity for Correspondences

We want to talk about continuity of correspondences analogous to continuity of functions. What should continuity mean?

We will discuss three main notions of continuity for correspondences, each of which can be motivated by thinking about what continuity means for a function $f : \mathbf{R}^n \to \mathbf{R}$.

Continuity for Correspondences

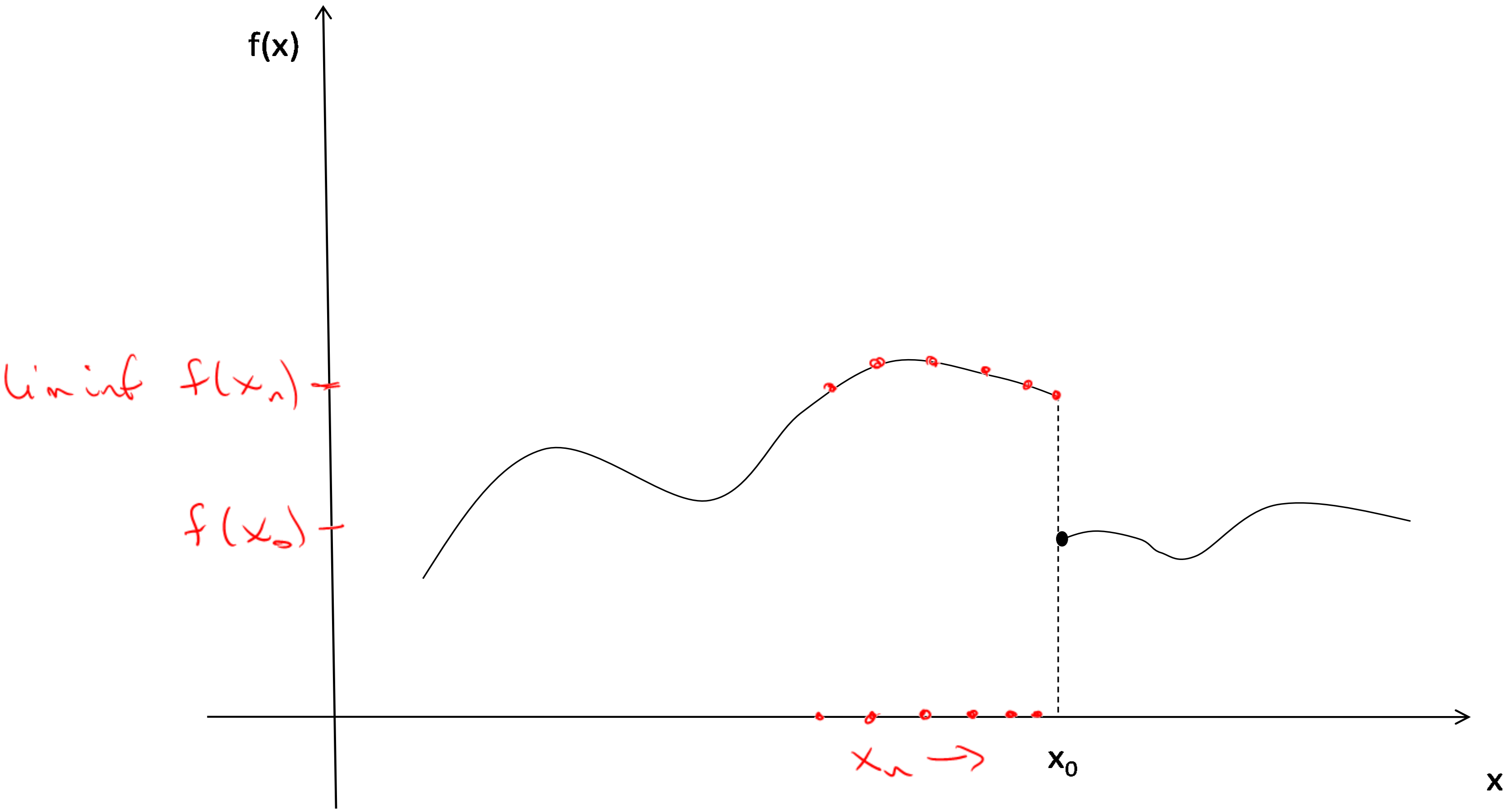
One way a function $f : \mathbf{R}^n \to \mathbf{R}$ may be discontinuous at a point x_0 is that it "jumps downward at the limit:"

$$\exists x_n \to x_0 \text{ s.t. } f(x_0) < \liminf f(x_n)$$

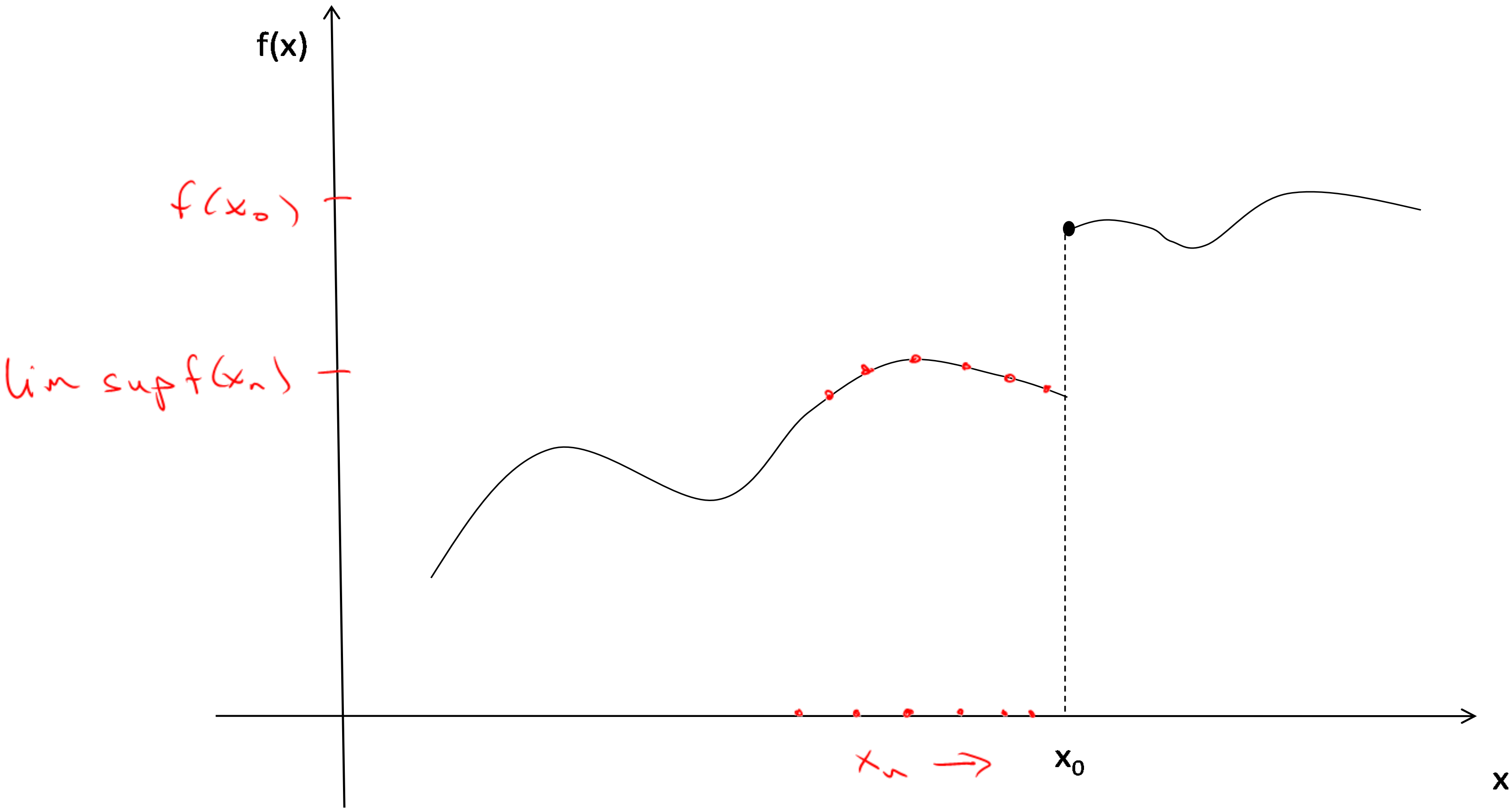
It could also "jump upward at the limit:"

$$\exists x_n \to x_0 \text{ s.t. } f(x_0) > \limsup f(x_n)$$

In either case, it doesn't matter whether the sequence x_n approaches x_0 from the left or the right (or both).



f "jumps downward at the limit" at x_0



f "jumps upward at the limit" at x_0

Continuity for Correspondences

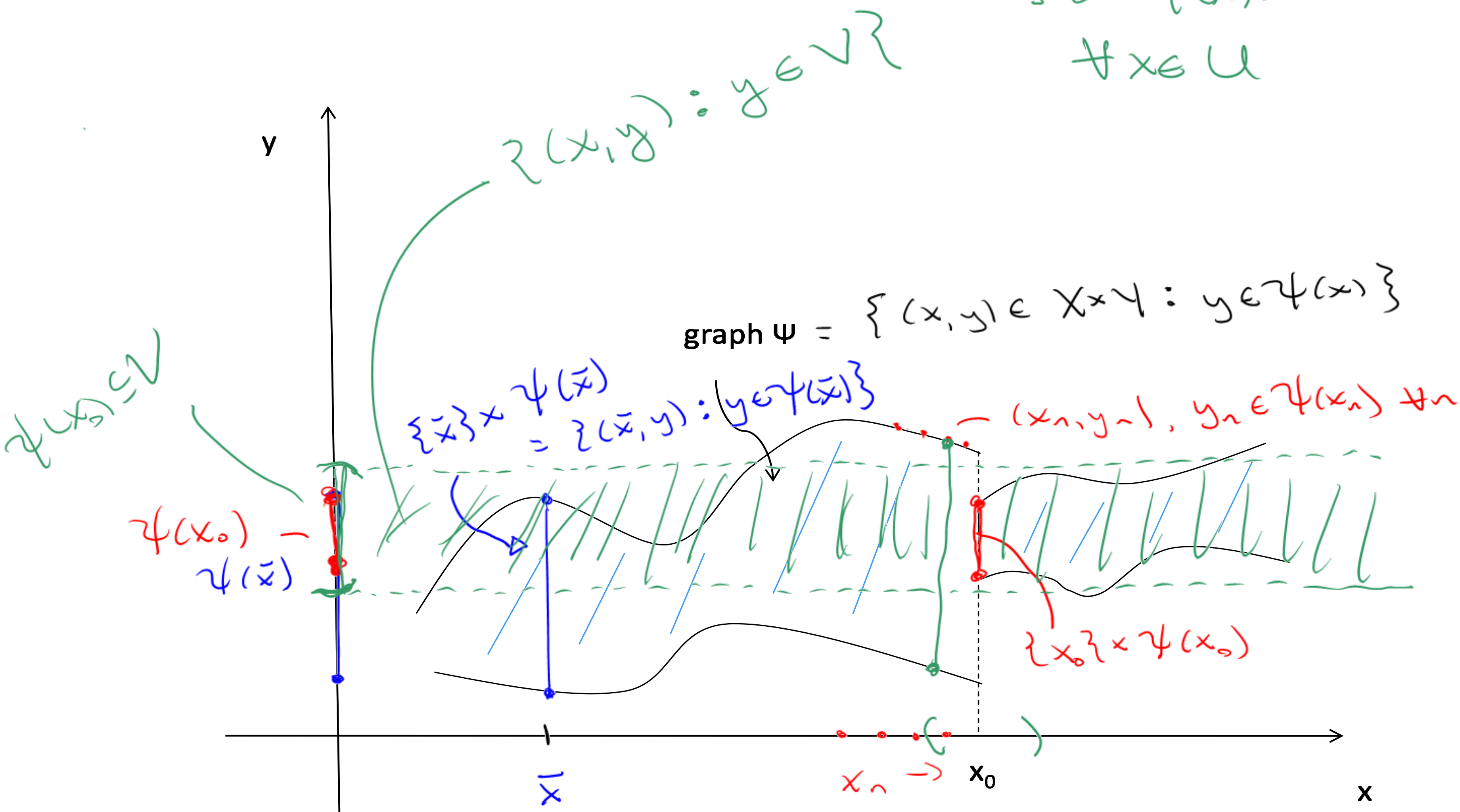
What should it mean for a *set* to "jump down" at the limit x_0?

It should mean the set suddenly gets smaller – it "implodes in the limit" – that is, there is a sequence $x_n \to x_0$ and points $y_n \in \Psi(x_n)$ that are far from every point of $\Psi(x_0)$ as $n \to \infty$.

$$? \quad \lim_{n} \psi(x_n)$$

$$\psi(x_0) \cup \bigcup_{n} \psi(x_n)$$

Ψ is not uhc at x_0 : $\nexists$ open $U \ni x_0$ s.t. $\Psi(x) \subseteq V$ $\forall x \in U$



$x_n \to x_0$, $y_n \in \Psi(x_n)$ $\forall n$ but y_n far from every point of $\Psi(x_n)$ as $n \to \infty$

Ψ "implodes in the limit" at x_0

Continuity for Correspondences

Similarly, what should it mean for a set to "jump up" at the limit?

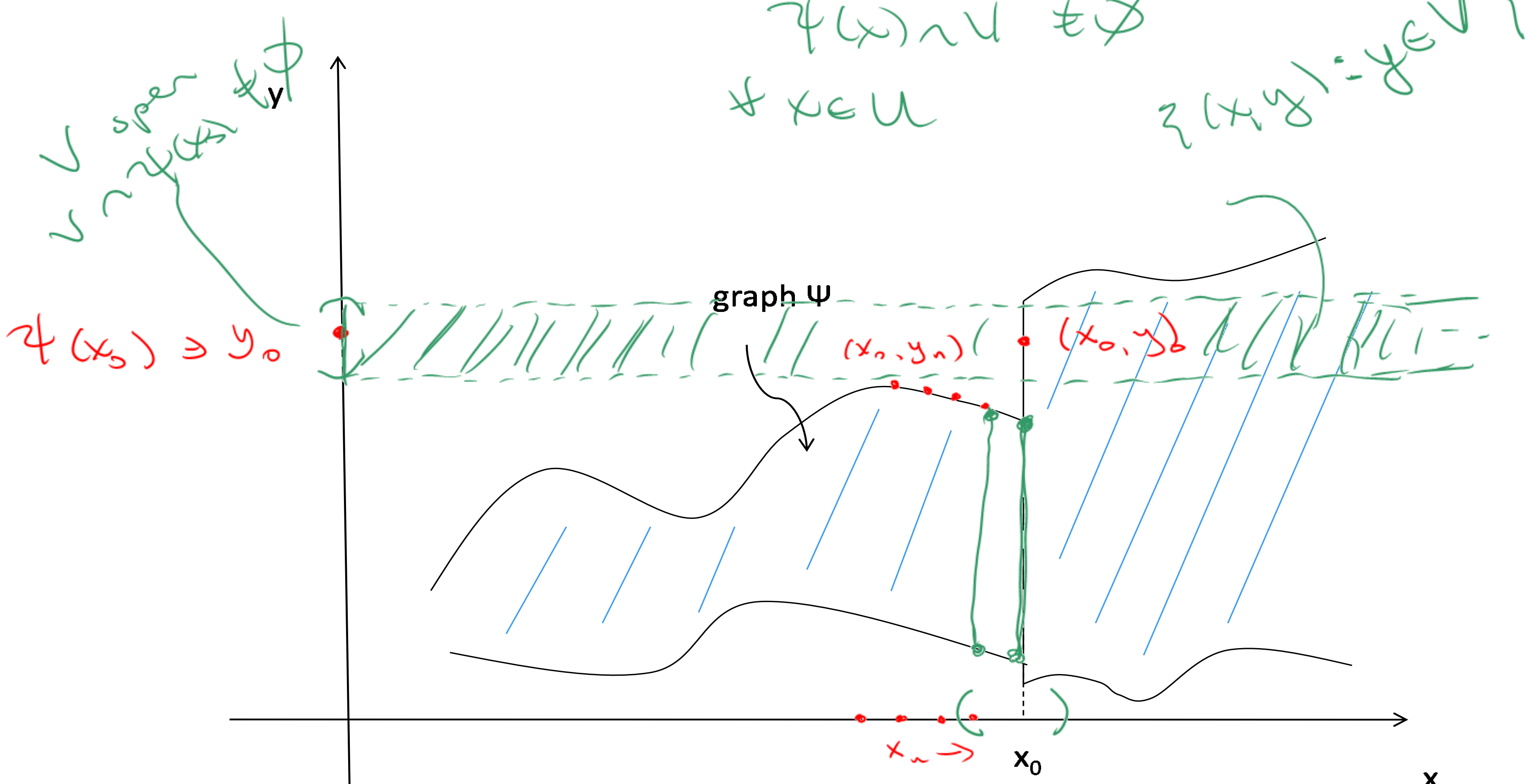
This should mean that that the set suddenly gets bigger – it "explodes in the limit" – that is, there is a point y in $\Psi(x_0)$ and a sequence $x_n \to x_0$ such that y is far from every point of $\Psi(x_n)$ as $n \to \infty$.

Ψ is not lhc at x_0:

$\nexists$ open $U \ni x_0$ s.t.

$\Psi(x) \cap V \neq \emptyset$

$\forall x \in U$



$y \in \Psi(x_0)$, $x_n \to x_0$ but y far from all elements of $\Psi(x_n)$ as $n \to \infty$

Ψ "explodes in the limit" at x_0

Continuity for Correspondences

Definition 3. *Let* $X \subseteq \mathbf{E}^n$*,* $Y \subseteq \mathbf{E}^m$*, and* $\Psi : X \to 2^Y$*.*

- Ψ *is* upper hemicontinuous (uhc) *at* $x_0 \in X$ *if, for every open set* $V \supseteq \Psi(x_0)$*, there is an open set* U *with* $x_0 \in U$ *such that*

$$\Psi(x) \subseteq V \text{ for every } x \in U \cap X$$

- Ψ *is* lower hemicontinuous (lhc) *at* $x_0 \in X$ *if, for every open set* V *such that* $\Psi(x_0) \cap V \neq \emptyset$*, there is an open set* U *with* $x_0 \in U$ *such that*

$$\Psi(x) \cap V \neq \emptyset \text{ for every } x \in U \cap X$$

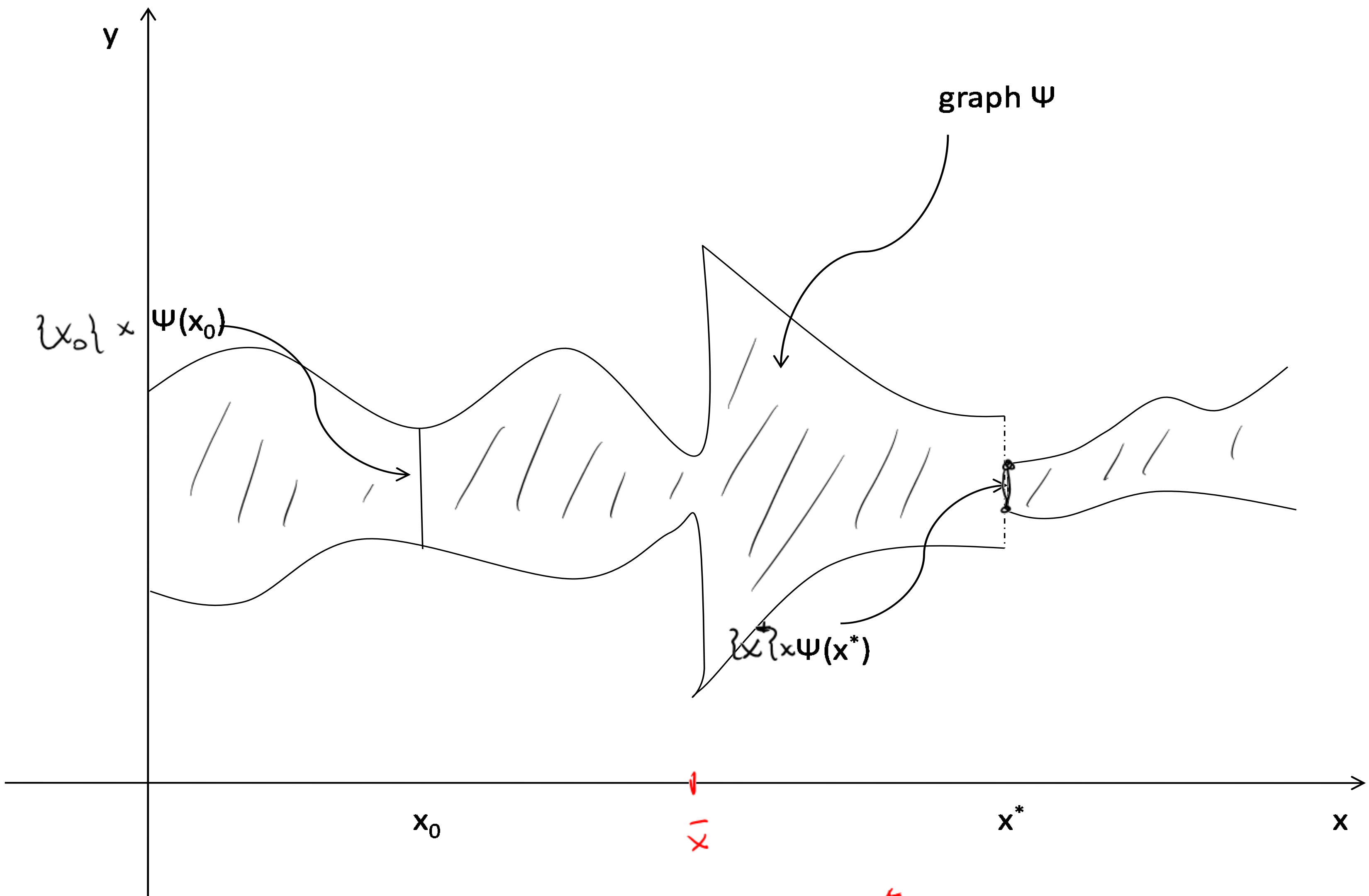
- Ψ *is* continuous *at* $x_0 \in X$ *if it is both uhc and lhc at* x_0*.*

- Ψ *is* upper hemicontinuous *(respectively* lower hemicontinuous, continuous*) if it is uhc (respectively lhc, continuous) at every* $x \in X$*.*

Continuity for Correspondences

Upper hemicontinuity reflects the requirement that Ψ doesn't "implode in the limit" at x_0; lower hemicontinuity reflects the requirement that Ψ doesn't "explode in the limit" at x_0.

Notice that upper and lower hemicontinuity are not nested: a correspondence can be upper hemicontinuous but not lower hemicontinuous, or lower hemicontinuous but not upper hemicontinuous.



Ψ not uhc at x^*

Ψ not lhc at $\bar{x}$

Continuity for Correspondences

An alternative notion of continuity looks instead at properties of the graph of the correspondence.

The *graph* of a correspondence $\Psi : X \to 2^Y$ is the set

$$\text{graph } \Psi = \{(x, y) \in X \times Y : y \in \Psi(x)\}$$

graph $f = \{(x, y) : y = f(x)\}$

Continuity for Correspondences

Recall that a function $f : \mathbf{R}^n \to \mathbf{R}$ is continuous if and only if whenever $x_n \to x$, $f(x_n) \to f(x)$. We can translate this into a statement about its graph.

Suppose $\{(x_n, y_n)\} \subseteq$ (graph f) and $(x_n, y_n) \to (x, y)$. Since f is a function, $(x_n, y_n) \in$ (graph f) $\iff y_n = f(x_n)$.

$$\text{So } f \text{ is continuous} \quad \Rightarrow \quad y = \lim y_n = \lim f(x_n) = f(x)$$
$$\Rightarrow \quad (x, y) \in (\text{graph } f)$$

So if f is continuous then each convergent sequence $\{(x_n, y_n)\}$ in graph f converges to a point (x, y) in graph f, that is, graph f is closed.

Closed Graph

Definition 4. *Let* $X \subseteq \mathbf{E}^n$, $Y \subseteq \mathbf{E}^m$. *A correspondence* $\Psi : X \to 2^Y$ *has* closed graph *if its graph is a closed subset of* $X \times Y$, *that is, if for any sequences* $\{x_n\} \subseteq X$ *and* $\{y_n\} \subseteq Y$ *such that* $x_n \to x \in X$, $y_n \to y \in Y$ *and* $y_n \in \Psi(x_n)$ *for each* n, *then* $y \in \Psi(x)$.

Closed Graph

Example: Consider the correspondence $\Psi : [0,1] \to 2^{\mathbb{R}}$

$$\Psi(x) = \begin{cases} \left\{\frac{1}{x}\right\} & \text{if } x \in (0,1] \\ \{0\} & \text{if } x = 0 \end{cases}$$

Ψ is not uhc at 0:

Let $V = (-0.1, 0.1)$. Then $\Psi(0) = \{0\} \subset V$, but no matter how close x is to 0,

$$\Psi(x) = \left\{\frac{1}{x}\right\} \not\subseteq V \qquad \forall x \in (0,1]$$

so Ψ is not uhc at 0. However, note that Ψ has closed graph.

$\{(x_n, y_n)\} \subseteq \text{graph } \psi$

and $(x_n, y_n) \to (x, y)$

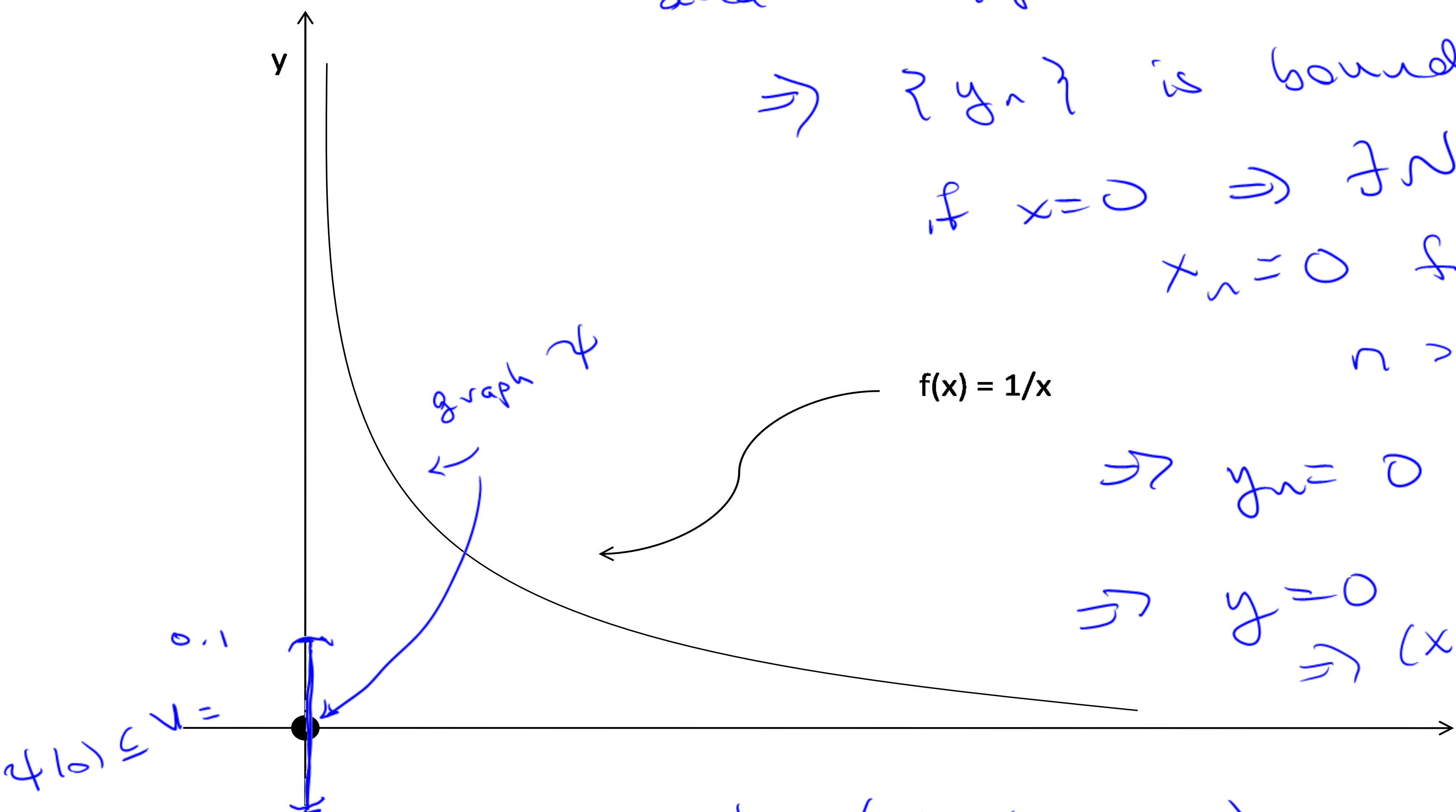
$\Rightarrow \{y_n\}$ is bounded

if $x = 0 \Rightarrow \exists N$ s.t. $x_n = 0$ for all $n > N$

$\Rightarrow y_n = 0 \ \forall n > N$

$\Rightarrow y = 0$

$\Rightarrow (x, y) = (0, 0) \in \text{graph } \psi$



$\forall x > 0 \ \psi(x) \notin V = (-0.1, 0.1)$

$(x_n, y_n) \in \text{graph } \Psi \quad \forall n$

$(x_n, y_n) \to (x, y) \qquad \{y_n\}$ is bounded

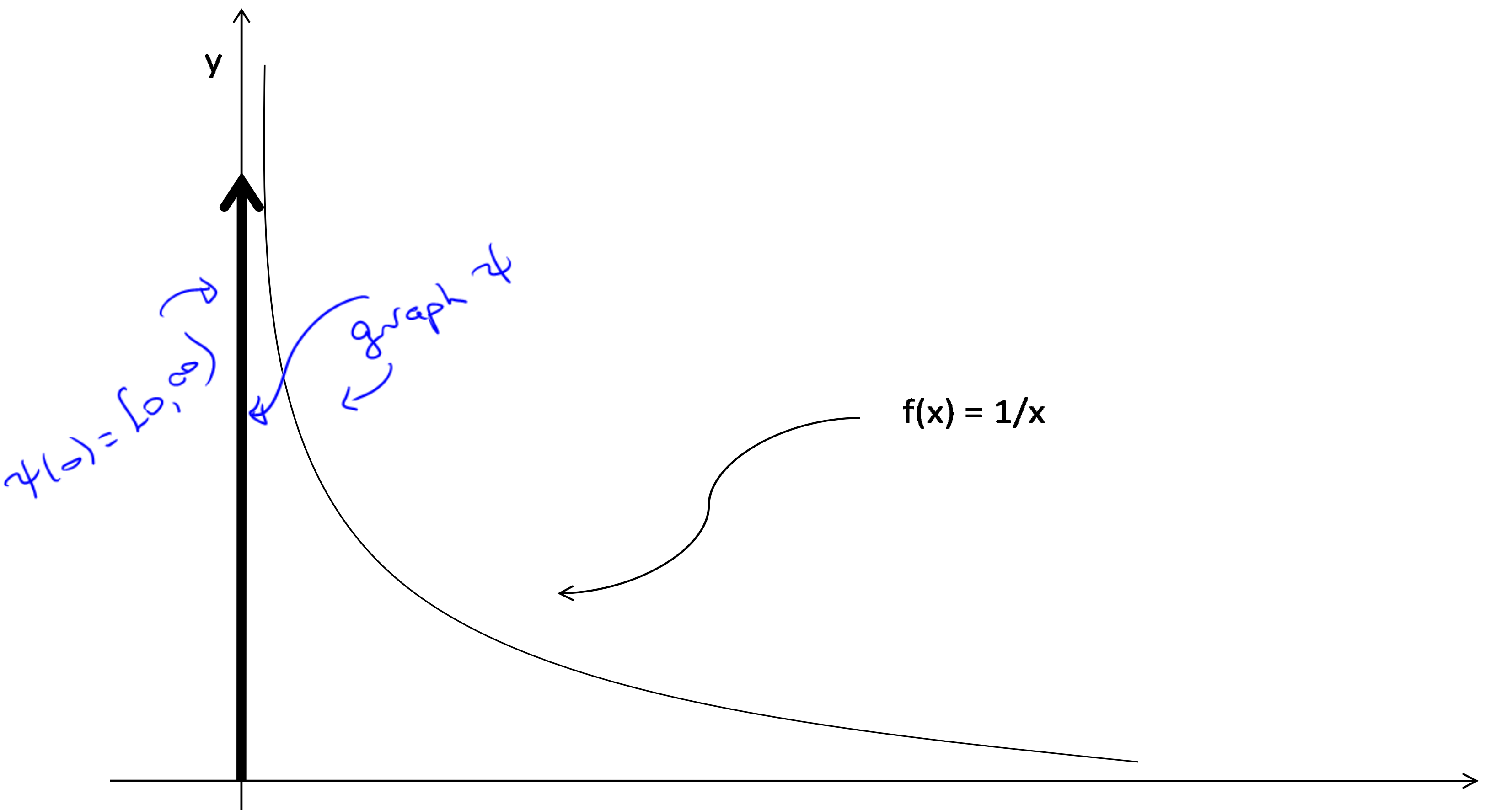
$\underline{x = 0}$: $\exists N$ s.t. $x_n = 0 \quad \forall n > N$

Continuity for Correspondences

Example: Consider the correspondence $\Psi: [0,1] \to 2^{\mathbb{R}}$

$$\Psi(x) = \begin{cases} \left\{\frac{1}{x}\right\} & \text{if } x \in (0,1] \\ \mathbf{R}_+ & \text{if } x = 0 \end{cases}$$

$\Psi(0) = [0, \infty)$, and $\Psi(x) \subseteq \Psi(0)$ for every $x \in [0,1]$. So if $V \supseteq \Psi(0)$ then $V \supseteq \Psi(x)$ for all x. Thus, Ψ is uhc, and has closed graph.



$\psi(0) = [0,\infty) \subseteq V \Rightarrow \forall \varepsilon > 0 \quad \psi(x) = \{\frac{1}{x}\} \subseteq V \quad \forall x \in (0,\varepsilon)$

ψ is uhc and has closed graph

Upper Hemicontinuity and Closed Graph

For a function, upper hemi-continuity and continuity coincide.

Theorem 3. *Let $X \subseteq \mathbf{E}^n$, $Y \subseteq \mathbf{E}^m$ and $f : X \to Y$. Let $\Psi : X \to 2^Y$ be the correspondence given by $\Psi(x) = \{f(x)\}$ for all $x \in X$. Then Ψ is uhc if and only if f is continuous.*

Proof. We consider the metric spaces (X, d) and (Y, d), where d is the Euclidean metric. Fix V open in Y. Then

$$\begin{aligned} f^{-1}(V) &= \{x \in X : f(x) \in V\} \\ &= \{x \in X : \Psi(x) \subseteq V\} \end{aligned}$$

Thus, f is continuous if and only if $f^{-1}(V)$ is open in X for each open V in Y, if and only if $\{x \in X : \Psi(x) \subseteq V\}$ is open in X for each open V in Y, if and only if Ψ is uhc (as an exercise, think through why this last equivalence holds). $\square$

- same result holds for lhc (exercise)

Continuity for Correspondences

For a general correspondence, these notions are not nested:

- A closed graph correspondence need not be uhc, as the first example above illustrates.

- Conversely an uhc correspondence need not have closed graph, or even have closed values.

Continuity for Correspondences

Definition 5. *A correspondence* $\Psi : X \to 2^Y$ *is called* closed-valued *if* $\Psi(x)$ *is a closed subset of* Y *for all* x*;* Ψ *is called* compact-valued *if* $\Psi(x)$ *is compact for all* x*.*

For closed-valued correspondences these concepts can be more tightly connected.

- A closed-valued and upper hemicontinuous correspondence must have closed graph.

- For a closed-valued correspondence with a compact range, upper hemicontinuity is equivalent to closed graph.

Upper Hemicontinuity and Closed Graph

Theorem 4 (Not in de la Fuente)**.** *Suppose* $X \subseteq \mathbf{E}^n$ *and* $Y \subseteq \mathbf{E}^m$*, and* $\Psi : X \to 2^Y$*.*

(i) If Ψ *is closed-valued and uhc, then* Ψ *has closed graph.*

(ii) If Ψ *has closed graph and there is an open set* W *with* $x_0 \in W$ *and a compact set* Z *such that* $x \in W \cap X \Rightarrow \Psi(x) \subseteq Z$*, then* Ψ *is uhc at* x_0*.*

(iii) If Y *is compact, then* Ψ *has closed graph* $\iff$ Ψ *is closed-valued and uhc.*

Proof. (i) Suppose Ψ is closed-valued and uhc. If Ψ does not have closed graph, we can find a sequence $(x_n, y_n) \to (x_0, y_0)$, where (x_n, y_n) lies in the graph of Ψ (so $y_n \in \Psi(x_n)$) but (x_0, y_0) does not lie in the graph of Ψ (so $y_0 \notin \Psi(x_0)$). Since Ψ is closed-valued, $\Psi(x_0)$ is closed. Since $y_0 \notin \Psi(x_0)$, there is some $\varepsilon > 0$ such that

$$\Psi(x_0) \cap B_{2\varepsilon}(y_0) = \emptyset$$

so

$$\Psi(x_0) \subseteq \mathbf{E}^m \setminus B_\varepsilon[y_0] \quad \longleftarrow \text{open}$$

Let $V = \mathbf{E}^m \setminus B_\varepsilon[y_0]$. Then V is open, and $\Psi(x_0) \subseteq V$. Since Ψ is uhc, there is an open set U with $x_0 \in U$ such that

$$x \in U \cap X \Rightarrow \Psi(x) \subseteq V$$

Since $(x_n, y_n) \to (x_0, y_0)$, $x_n \in U$ for n sufficiently large, so

$$y_n \in \Psi(x_n) \subseteq V \quad = \mathbb{R}^m \setminus B_\varepsilon[y_0]$$

Thus for n sufficiently large, $\|y_n - y_0\| \geq \varepsilon$, which implies that $y_n \not\to y_0$, and $(x_n, y_n) \not\to (x_0, y_0)$, a contradiction. Thus Ψ is closed-graph.

(ii) Now, suppose Ψ has closed graph and there is an open set W with $x_0 \in W$ and a compact set Z such that

$$x \in W \cap X \Rightarrow \Psi(x) \subseteq Z$$

Since Ψ has closed graph, it is closed-valued. Let V be any open set such that $V \supseteq \Psi(x_0)$. We need to show there exists an open set U with $x_0 \in U$ such that

$$x \in U \cap X \Rightarrow \Psi(x) \subseteq V$$

If not, we can find a sequence $x_n \to x_0$ and $y_n \in \Psi(x_n)$ such that $y_n \notin V \ \forall n$. Since $x_n \to x_0$, $x_n \in W \cap X$ for all n sufficiently large, and thus $\Psi(x_n) \subseteq Z$ for n sufficiently large. Since Z is compact, we can find a convergent subsequence $y_{n_k} \to y'$. Then

$$(x_{n_k}, y_{n_k}) \to (x_0, y')$$

$y_{n_k} \to y'$

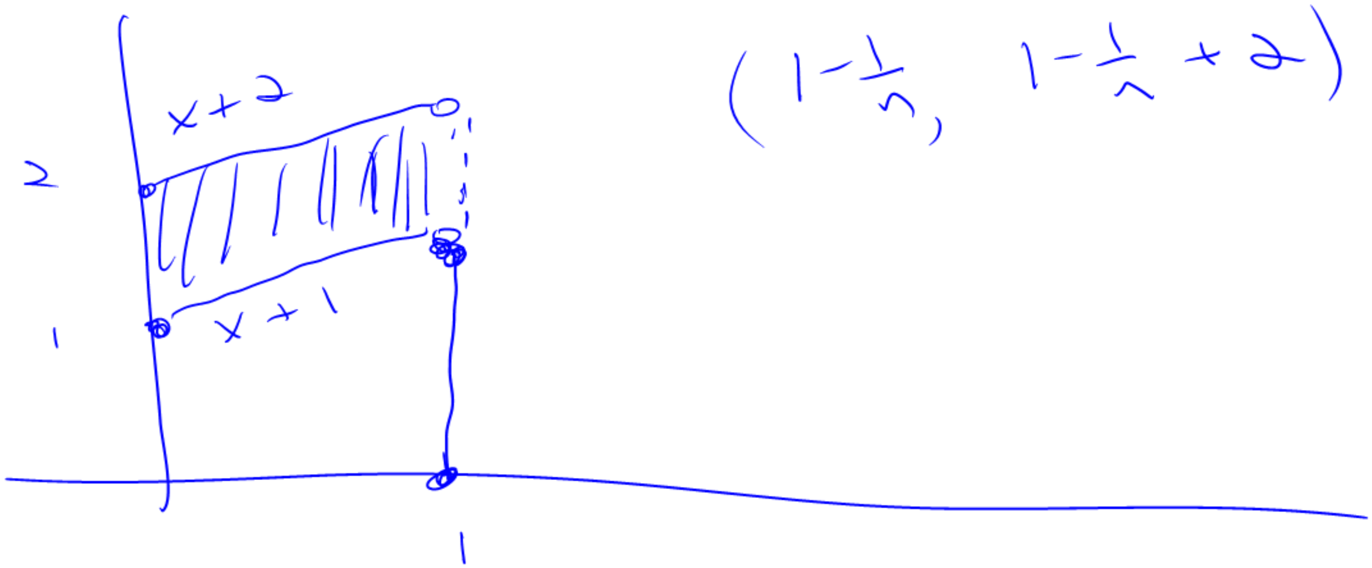
Since Ψ has closed graph, $y' \in \Psi(x_0)$, so $y' \in V$. Since V is open, $y_{n_k} \in V$ for all k sufficiently large, a contradiction. Thus, Ψ is uhc at x_0.

(iii) Follows from (i) and (ii). $\square$

$\psi : [0,1] \to 2^{\mathbb{R}}$

$$\psi(x) = \begin{cases} [x+1, x+2] & x \in [0,1) \\ [0,1] & x = 1 \end{cases}$$

$\psi(x)$ closed $\forall x \in [0,1]$



$\left(1 - \frac{1}{n},\ 1 - \frac{1}{n} + 2\right)$

Sequential Characterizations

Upper and lower hemicontinuity can be given sequential characterizations that are useful in applications.

Theorem 5 (Thm. 11.2)**.** *Suppose* $X \subseteq \mathbf{E}^n$ *and* $Y \subseteq \mathbf{E}^m$. *A compact-valued correspondence* $\Psi : X \to 2^Y$ *is uhc at* $x_0 \in X$ *if and only if, for every sequence* $\{x_n\} \subseteq X$ *with* $x_n \to x_0$*, and every sequence* $\{y_n\}$ *such that* $y_n \in \Psi(x_n)$ *for every* n*, there is a convergent subsequence* $\{y_{n_k}\}$ *such that* $\lim y_{n_k} \in \Psi(x_0)$*.*

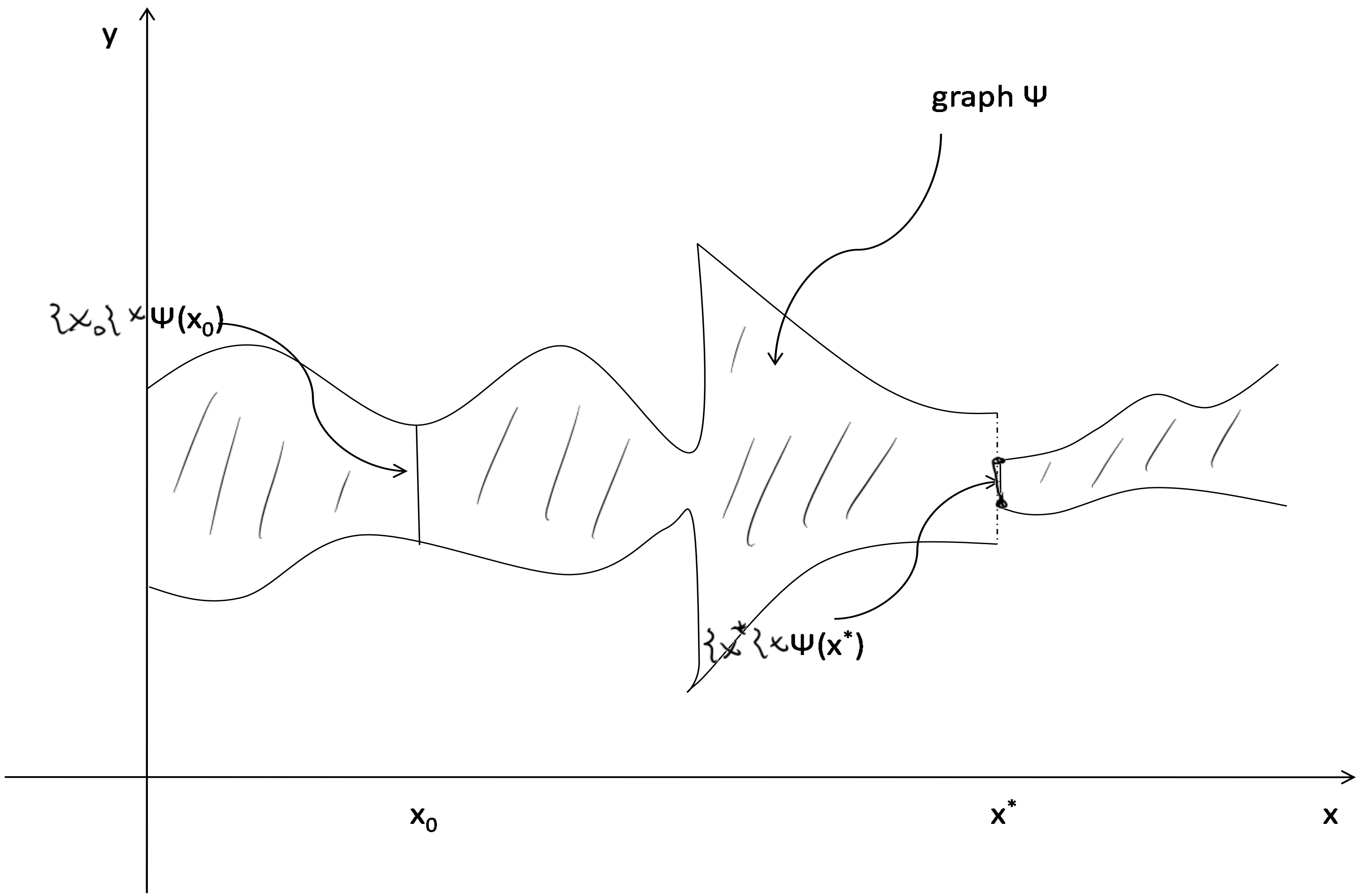
Proof. See de la Fuente. $\square$

Note that this characterization of upper hemicontinuity requires the correspondence to have compact values.

Sequential Characterizations

Theorem 6 (Thm. 11.3)**.** *A correspondence* $\Psi : X \to 2^Y$ *is lhc at* $x_0 \in X$ *if and only if, for every sequence* $\{x_n\} \subseteq X$ *with* $x_n \to x_0$*, and every* $y_0 \in \Psi(x_0)$*, there exists a companion sequence* $\{y_n\}$ *with* $y_n \in \Psi(x_n)$ *for every* n *such that* $y_n \to y_0$*.*

Proof. See de la Fuente. $\square$


y
graph Ψ
{x₀}×Ψ(x₀)
{x*}×Ψ(x*)
x₀
x*
x