Econ 204 2020

Lecture 8

Outline

1. Bases
2. Linear Transformations
3. Isomorphisms
Linear Combinations and Spans

**Definition 1.** Let $X$ be a vector space over a field $F$. A linear combination of $x_1, \ldots, x_n \in X$ is a vector of the form

$$y = \sum_{i=1}^{n} \alpha_i x_i$$

where $\alpha_1, \ldots, \alpha_n \in F$.

$\alpha_i$ is the coefficient of $x_i$ in the linear combination.

If $V \subseteq X$, the span of $V$, denoted $\text{span} \, V$, is the set of all linear combinations of elements of $V$.

A set $V \subseteq X$ spans $X$ if $\text{span} \, V = X$. 
Linear Dependence and Independence

**Definition 2.** A set \( V \subseteq X \) is linearly dependent if there exist \( v_1, \ldots, v_n \in V \) and \( \alpha_1, \ldots, \alpha_n \in F \) not all zero such that

\[
\sum_{i=1}^{n} \alpha_i v_i = 0
\]

A set \( V \subseteq X \) is linearly independent if it is not linearly dependent.

Thus \( V \subseteq X \) is linearly independent if and only if

\[
\sum_{i=1}^{n} \alpha_i v_i = 0, \quad v_i \in V \quad \forall i \Rightarrow \alpha_i = 0 \quad \forall i
\]
Bases

Definition 3. A Hamel basis (often just called a basis) of a vector space $X$ is a linearly independent set of vectors in $X$ that spans $X$.

Example: $\{(1,0),(0,1)\}$ is a basis for $\mathbb{R}^2$ (this is the standard basis).

$\alpha, \beta \in \mathbb{R}$:

$\alpha (1,0) + \beta (0,1) = (\alpha,\beta)$
Example, cont: \{ (1, 1), (-1, 1) \} is another basis for \( \mathbb{R}^2 \):

Suppose \((x, y) = \alpha (1, 1) + \beta (-1, 1)\) for some \(\alpha, \beta \in \mathbb{R}\)

\[
\begin{align*}
x &= \alpha - \beta \\
y &= \alpha + \beta \\
x + y &= 2\alpha \\
\Rightarrow \alpha &= \frac{x + y}{2} \\
y - x &= 2\beta \\
\Rightarrow \beta &= \frac{y - x}{2}
\end{align*}
\]

\((x, y) = \frac{x + y}{2} (1, 1) + \frac{y - x}{2} (-1, 1)\)

Since \((x, y)\) is an arbitrary element of \(\mathbb{R}^2\), \{ (1, 1), (-1, 1) \} spans \(\mathbb{R}^2\). If \((x, y) = (0, 0)\),

\[
\begin{align*}
\alpha &= \frac{0 + 0}{2} = 0, \\
\beta &= \frac{0 - 0}{2} = 0
\end{align*}
\]
so the coefficients are all zero, so \{(1,1),(-1,1)\} is linearly independent. Since it is linearly independent and spans \(\mathbb{R}^2\), it is a basis.

**Example:** \{\((1,0,0),(0,1,0)\)\} is not a basis of \(\mathbb{R}^3\), because it does not span \(\mathbb{R}^3\). 

**Example:** \{\((1,0),(0,1),(1,1)\)\} is not a basis for \(\mathbb{R}^2\).

\[
1(1,0) + 1(0,1) + (-1)(1,1) = (0,0)
\]

so the set is not linearly independent.
Bases

**Theorem 1** (Thm. 1.2'). *Let $V$ be a Hamel basis for $X$. Then every vector $x \in X$ has a unique representation as a linear combination of a finite number of elements of $V$ (with all coefficients nonzero).*

**Proof.** Let $x \in X$. Since $V$ spans $X$, we can write

$$x = \sum_{s \in S_1} \alpha_s v_s$$

where $S_1$ is finite, $\alpha_s \in F$, $\alpha_s \neq 0$, and $v_s \in V$ for each $s \in S_1$. Now, suppose

$$x = \sum_{s \in S_1} \alpha_s v_s = \sum_{s \in S_2} \beta_s v_s$$

*The unique representation of 0 is $0 = \sum_{i \in \emptyset} \alpha_i b_i$.}
where $S_2$ is finite, $\beta_s \in F$, $\beta_s \neq 0$, and $v_s \in V$ for each $s \in S_2$. Let $S = S_1 \cup S_2$, and define

$$\alpha_s = 0 \text{ for } s \in S_2 \setminus S_1$$
$$\beta_s = 0 \text{ for } s \in S_1 \setminus S_2$$

Then

$$0 = x - x$$
$$= \sum_{s \in S_1} \alpha_s v_s - \sum_{s \in S_2} \beta_s v_s$$
$$= \sum_{s \in S} \alpha_s v_s - \sum_{s \in S} \beta_s v_s$$
$$= \sum_{s \in S} (\alpha_s - \beta_s) v_s$$

Since $V$ is linearly independent, we must have $\alpha_s - \beta_s = 0$, so $\alpha_s = \beta_s$, for all $s \in S$.

$$s \in S_1 \iff \alpha_s \neq 0 \iff \beta_s \neq 0 \iff s \in S_2$$
so $S_1 = S_2$ and $\alpha_s = \beta_s$ for $s \in S_1 = S_2$, so the representation is unique. \qed
Bases

**Theorem 2.** *Every vector space has a Hamel basis.*

*Proof.* The proof uses the Axiom of Choice. Indeed, the theorem is equivalent to the Axiom of Choice. 

□
Bases

A closely related result, from which you can derive the previous result, shows that any linearly independent set $V$ in a vector space $X$ can be extended to a basis of $X$.

**Theorem 3.** *If $X$ is a vector space and $V \subseteq X$ is linearly independent, then there exists a linearly independent set $W \subseteq X$ such that* 

$$V \subseteq W \subseteq \text{span } W = X$$
Bases

**Theorem 4.** *Any two Hamel bases of a vector space* \( X \) *have the same cardinality (are numerically equivalent).*

**Proof.** The proof depends on the so-called Exchange Lemma, whose idea we sketch. Suppose that \( V = \{v_\lambda : \lambda \in \Lambda\} \) and \( W = \{w_\gamma : \gamma \in \Gamma\} \) are Hamel bases of \( X \). Remove one vector \( v_{\lambda_0} \) from \( V \), so that it no longer spans (if it did still span, then \( v_{\lambda_0} \) would be a linear combination of other elements of \( V \), and \( V \) would not be linearly independent). If \( w_\gamma \in \text{span} \left( V \setminus \{v_{\lambda_0}\} \right) \) for every \( \gamma \in \Gamma \), then since \( W \) spans, \( V \setminus \{v_{\lambda_0}\} \) would also span, contradiction. Thus, we can choose \( \gamma_0 \in \Gamma \) such that

\[
w_{\gamma_0} \notin \text{span} \left( V \setminus \{v_{\lambda_0}\} \right)
\]
Because \( w_{\gamma_0} \in \text{span} \ V \), we can write

\[
  w_{\gamma_0} = \sum_{i=0}^{n} \alpha_i v_{\lambda_i}
\]

where \( \alpha_0 \), the coefficient of \( v_{\lambda_0} \), is not zero (if it were, then we would have \( w_{\gamma_0} \in \text{span} \ (V \setminus \{v_{\lambda_0}\}) \)). Since \( \alpha_0 \neq 0 \), we can solve for \( v_{\lambda_0} \) as a linear combination of \( w_{\gamma_0} \) and \( v_{\lambda_1}, \ldots, v_{\lambda_n} \), so

\[
  \text{span} \left( \left( V \setminus \{v_{\lambda_0}\} \right) \cup \{w_{\gamma_0}\} \right) \supseteq \text{span} V = \text{span} \left( V \setminus \{v_{\lambda_0}\} \right) \cup \{v_{\lambda_0}\}
\]

\[
= X
\]

so

\[
\left( \left( V \setminus \{v_{\lambda_0}\} \right) \cup \{w_{\gamma_0}\} \right)
\]

spans \( X \). From the fact that \( w_{\gamma_0} \not\in \text{span} \ (V \setminus \{v_{\lambda_0}\}) \) one can
show that

\[ ((V \setminus \{v_{\lambda_0}\}) \cup \{w_{\gamma_0}\}) \]

is linearly independent, so it is a basis of \( X \). Repeat this process to exchange every element of \( V \) with an element of \( W \) (when \( V \) is uncountable, this is done by a process called transfinite induction). At the end, we obtain a bijection from \( V \) to \( W \), so that \( V \) and \( W \) are numerically equivalent. \( \square \)
Dimension

Definition 4. The dimension of a vector space $X$, denoted $\dim X$, is the cardinality of any basis of $X$.

For $V \subseteq X$, $|V|$ denotes the cardinality of the set $V$.

- If $\dim X = n$ for some $n \in \mathbb{N}$,
  $X$ is finite-dimensional.

  Otherwise, $X$ is infinite-dimensional.
Dimension

Example: The set of all $m \times n$ real-valued matrices is a vector space over $\mathbb{R}$. A basis is given by

$$\{E_{ij} : 1 \leq i \leq m, 1 \leq j \leq n\}$$

where

$$\left(E_{ij}\right)_{k\ell} = \begin{cases} 1 & \text{if } k = i \text{ and } \ell = j \\ 0 & \text{otherwise}. \end{cases}$$

The dimension of the vector space of $m \times n$ matrices is $mn$. 

$E_{ij} = \begin{pmatrix} 0 & \ldots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \ldots & 0 \end{pmatrix}$
Theorem 5 (Thm. 1.4). Suppose $\dim X = n \in \mathbb{N}$. If $V \subseteq X$ and $|V| > n$, then $V$ is linearly dependent.

If not, $V$ is linearly independent, so $V$ can be extended to a basis $W$ of $X$, and

$$V \subseteq W \implies n < |V| \leq |W|$$

contradiction.
Dimension and Dependence

Theorem 6 (Thm. 1.5'). Suppose \( \dim X = n \in \mathbb{N} \), \( V \subseteq X \), and \( |V| = n \).

- If \( V \) is linearly independent, then \( V \) spans \( X \), so \( V \) is a Hamel basis.

- If \( V \) spans \( X \), then \( V \) is linearly independent, so \( V \) is a Hamel basis.

1. Otherwise, extend \( V \) to a basis \( W \) of \( X \), with \( V \neq W \), so \( |W| > |V| = n \), contradiction.
2. Otherwise, choose \( V' \neq V \) a basis for \( X \), and \( |V'| < |V| = n \), contradiction.
Linear Transformations

Definition 5. Let $X$ and $Y$ be two vector spaces over the field $F$. We say $T : X \rightarrow Y$ is a linear transformation if

$$T(\alpha_1 x_1 + \alpha_2 x_2) = \alpha_1 T(x_1) + \alpha_2 T(x_2) \quad \forall x_1, x_2 \in X, \alpha_1, \alpha_2 \in F$$

Let $L(X,Y)$ denote the set of all linear transformations from $X$ to $Y$.

Equivalently:

- $T(\alpha x) = \alpha T(x) \quad \forall \alpha \in F, \forall x \in X$
- $T(x_1 + x_2) = T(x_1) + T(x_2) \quad \forall x_1, x_2 \in X$
Linear Transformations

**Theorem 7.** \( L(X, Y) \) is a vector space over \( F \).

**Proof.** First, define linear combinations in \( L(X, Y) \) as follows. For \( T_1, T_2 \in L(X, Y) \) and \( \alpha, \beta \in F \), define \( \alpha T_1 + \beta T_2 \) by

\[
(\alpha T_1 + \beta T_2)(x) = \alpha T_1(x) + \beta T_2(x)
\]

We need to show that \( \alpha T_1 + \beta T_2 \in L(X, Y) \).

\[
(\alpha T_1 + \beta T_2)(\gamma x_1 + \delta x_2) = \gamma (\alpha T_1(x_1) + \beta T_2(x_2)) + \delta (\alpha T_1(x_2) + \beta T_2(x_2))
\]

(definition)

\[
= \gamma (\alpha T_1(x_1) + \beta T_2(x_2)) + \delta (\alpha T_1(x_2) + \beta T_2(x_2))
\]

(T1, T2 linear)

(definition again)

\[
= \gamma (\alpha T_1 + \beta T_2)(x_1) + \delta (\alpha T_1 + \beta T_2)(x_2)
\]

(collect terms)
\[ + : L(X, Y) \times L(X, Y) \to L(X, Y) \]

\[ \cdot : \mathbb{F} \times L(X, Y) \to L(X, Y) \]

so \( \alpha T_1 + \beta T_2 \in L(X, Y) \).

The rest of the proof involves straightforward checking of the vector space axioms. \( \square \)
Compositions of Linear Transformations

Given $R \in L(X, Y)$ and $S \in L(Y, Z)$, $S \circ R : X \to Z$. We will show that $S \circ R \in L(X, Z)$, that is, the composition of two linear transformations is linear.

$$(S \circ R)(\alpha x_1 + \beta x_2) = S(R(\alpha x_1 + \beta x_2)) \quad (\text{defn of } S \circ R)$$

$$= S(\alpha R(x_1) + \beta R(x_2)) \quad (R \text{ linear})$$

$$= \alpha S(R(x_1)) + \beta S(R(x_2)) \quad (S \text{ linear})$$

$$= \alpha(S \circ R)(x_1) + \beta(S \circ R)(x_2) \quad (\text{defn of } S \circ R)$$

so $S \circ R \in L(X, Z)$. 

Kernel and Rank

Definition 6. Let $T \in L(X, Y)$.

- The image of $T$ is $\text{Im} T = T(X) \subseteq Y$
  - can show $\text{Im} T$ is a vector subspace of $Y$

- The kernel of $T$ is $\text{ker} T = \{x \in X : T(x) = 0\}$ (null space of $T$)

- The rank of $T$ is $\text{Rank} T = \dim(\text{Im} T)$

Recall:
- $W \subseteq X$ is a vector subspace if it is a vector space over $F$ under $+ , \cdot$ from $X$
- $W \subseteq X , W \neq \emptyset$ is a vector subspace if $\forall \alpha , \beta \in F \quad \forall w , w_1 , w_2 \in W \quad (\alpha w_1 + \beta w_2) \in W$
Rank-Nullity Theorem

**Theorem 8** (Thms. 2.9, 2.7, 2.6: The Rank-Nullity Theorem). *Let X be a finite-dimensional vector space, \( T \in L(X, Y) \). Then \( \text{Im } T \) and \( \ker T \) are vector subspaces of \( Y \) and \( X \) respectively, and

\[
\dim X = \dim \ker T + \text{Rank } T
\]

**nullity of \( T \)**

**Sketch:**

1. Show \( \text{Im } T, \ker T \) are vector subspaces.
2. Take \( \{v_1, \ldots, v_k\} \) a basis for \( \ker T \).
3. Extend to \( \{v_1, \ldots, v_k, w_1, \ldots, w_r\} \) a basis for \( X \).
4. Show \( \{T(w_1), \ldots, T(w_r)\} \) is a basis for \( \text{Im } T \).
Kernel and Rank

**Theorem 9** *(Thm. 2.13).* $T \in L(X, Y)$ is one-to-one if and only if $\ker T = \{0\}$.

\[ \implies \quad \text{Proof.} \text{ Suppose } T \text{ is one-to-one. Suppose } x \in \ker T. \text{ Then } T(x) = 0. \text{ But since } T \text{ is linear, } T(0) = T(0 \cdot 0) = 0 \cdot T(0) = 0. \text{ Since } T \text{ is one-to-one, } x = 0, \text{ so } \ker T = \{0\}. \]

\[ \Leftarrow \quad \text{Conversely, suppose that } \ker T = \{0\}. \text{ Suppose } T(x_1) = T(x_2). \text{ Then} \]

\[
T(x_1 - x_2) = T(x_1) - T(x_2) = 0
\]

which says $x_1 - x_2 \in \ker T$, so $x_1 - x_2 = 0$, so $x_1 = x_2$. Thus, $T$ is one-to-one. $\square$
Invertible Linear Transformations

**Definition 7.** $T \in L(X, Y)$ is invertible if there exists a function $S : Y \to X$ such that

\[
S(T(x)) = x \quad \forall x \in X \\
T(S(y)) = y \quad \forall y \in Y
\]

Denote $S$ by $T^{-1}$.

Note that $T$ is invertible if and only if it is one-to-one and onto. This is just the condition for the existence of an inverse function. The linearity of the inverse follows from the linearity of $T$.

(we will show this)
Invertible Linear Transformations

**Theorem 10** (Thm. 2.11). If $T \in L(X,Y)$ is invertible, then $T^{-1} \in L(Y,X)$, i.e. $T^{-1}$ is linear.

**Proof.** Suppose $\alpha, \beta \in F$ and $v, w \in Y$. Since $T$ is invertible, there exist unique $v', w' \in X$ such that

\[
T(v') = v \quad T^{-1}(v) = v' \\
T(w') = w \quad T^{-1}(w) = w'.
\]

Then

\[
T^{-1}(\alpha v + \beta w) = T^{-1}(\alpha T(v') + \beta T(w')) \quad \text{(definition)} \\
= T^{-1}(T(\alpha v' + \beta w')) \quad \text{(T linear)} \\
= \alpha v' + \beta w' \quad \text{(defn of T^{-1})} \\
= \alpha T^{-1}(v) + \beta T^{-1}(w) \quad \text{(defn of v', w')}
\]
so $T^{-1} \in L(Y, X)$. \hfill \Box
Linear Transformations and Bases

**Theorem 11** (Thm. 3.2). Let $X$ and $Y$ be two vector spaces over the same field $F$, and let $V = \{v_\lambda : \lambda \in \Lambda\}$ be a basis for $X$. Then a linear transformation $T \in L(X,Y)$ is completely determined by its values on $V$, that is:

1. Given any set $\{y_\lambda : \lambda \in \Lambda\} \subseteq Y$, $\exists T \in L(X,Y)$ s.t.
   \[ T(v_\lambda) = y_\lambda \quad \forall \lambda \in \Lambda \]

2. If $S, T \in L(X,Y)$ and $S(v_\lambda) = T(v_\lambda)$ for all $\lambda \in \Lambda$, then $S = T$. 
Proof. 1. If \( x \in X \), \( x \) has a unique representation of the form

\[
x = \sum_{i=1}^{n} \alpha_{i}v_{\lambda_{i}} \quad \alpha_{i} \neq 0 \quad i = 1, \ldots, n
\]

(Recall that if \( x = 0 \), then \( n = 0 \).) Define

\[
T(x) = \sum_{i=1}^{n} \alpha_{i}y_{\lambda_{i}} \quad \text{(so } T(v_{\lambda}) = y_{\lambda} \text{ for all } \lambda \text{ by defn)}
\]

Then \( T(x) \in Y \). The verification that \( T \) is linear is left as an exercise.
2. Suppose $S(v_\lambda) = T(v_\lambda)$ for all $\lambda \in \Lambda$. Given $x \in X$,

\[
S(x) = S\left( \sum_{i=1}^{n} \alpha_i v_{\lambda_i} \right) \\
= \sum_{i=1}^{n} \alpha_i S(v_{\lambda_i}) \quad \text{(S linear)} \\
= \sum_{i=1}^{n} \alpha_i T(v_{\lambda_i}) \quad \text{(S and T agree on \{v_\lambda : \lambda \in \Lambda\})} \\
= T\left( \sum_{i=1}^{n} \alpha_i v_{\lambda_i} \right) \quad \text{(T linear)} \\
= T(x)
\]

so $S = T$. \qed
Isomorphisms

Definition 8. Two vector spaces $X$ and $Y$ over a field $F$ are isomorphic if there is an invertible $T \in L(X,Y)$.

$T \in L(X,Y)$ is an isomorphism if it is invertible (one-to-one and onto).

Isomorphic vector spaces are essentially indistinguishable as vector spaces.
Isomorphisms

**Theorem 12** (Thm. 3.3). *Two vector spaces $X$ and $Y$ over the same field are isomorphic if and only if $\dim X = \dim Y$.***

$\Rightarrow$ : *Proof*. Suppose $X$ and $Y$ are isomorphic, and let $T \in L(X, Y)$ be an isomorphism. Let

$$U = \{u_\lambda : \lambda \in \Lambda\}$$

be a basis of $X$, and let $v_\lambda = T(u_\lambda)$ for each $\lambda \in \Lambda$. Set

$$V = \{v_\lambda : \lambda \in \Lambda\}$$

Since $T$ is one-to-one, $U$ and $V$ have the same cardinality. If $U = \{u_\lambda : \lambda \in \Lambda\}$,
\( y \in Y \), then there exists \( x \in X \) such that

\[
y = T(x)
\]

\[
y = T \left( \sum_{i=1}^{n} \alpha \lambda_i u \lambda_i \right)
\]

\[
y = \sum_{i=1}^{n} \alpha \lambda_i T(u \lambda_i)
\]

\[
y = \sum_{i=1}^{n} \alpha \lambda_i v \lambda_i
\]

(T is onto)

(linearity of \( T \))

(defn of \( v \lambda_i \))

which shows that \( V \) spans \( Y \). To see that \( V \) is linearly indepen-
dent, suppose

\[
0 = \sum_{i=1}^{m} \beta_i v \lambda_i \\
= \sum_{i=1}^{m} \beta_i T(\lambda_i) \\
= T \left( \sum_{i=1}^{m} \beta_i \lambda_i \right) \quad \text{(def of } v \lambda_i )
\]

Since \( T \) is one-to-one, \( \ker T = \{0\} \), so

\[
\sum_{i=1}^{m} \beta_i u \lambda_i = 0
\]

Since \( U \) is a basis, we have \( \beta_1 = \cdots = \beta_m = 0 \), so \( V \) is linearly independent. Thus, \( V \) is a basis of \( Y \); since \( U \) and \( V \) are numerically equivalent, \( \dim X = \dim Y \).

\[ |U| = |V| \]
Now suppose \( \dim X = \dim Y \). Let

\[
U = \{u_\lambda : \lambda \in \Lambda\} \quad \text{and} \quad V = \{v_\lambda : \lambda \in \Lambda\}
\]

be bases of \( X \) and \( Y \); note we can use the same index set \( \Lambda \) for both because \( \dim X = \dim Y \). By Theorem 3.2, there is a unique previous result
$T \in L(X, Y)$ such that $T(u_{\lambda}) = v_{\lambda}$ for all $\lambda \in \Lambda$. If $T(x) = 0$, then

\[ \Rightarrow \alpha_1 = \cdots = \alpha_n = 0 \text{ since } V \text{ is a basis} \]

\[ \Rightarrow x = 0 = \sum_{i=1}^{n} \alpha_i u_{\lambda_i} \]

\[ \Rightarrow \ker T = \{0\} \]

\[ \Rightarrow T \text{ is one-to-one} \]
If \( y \in Y \), write \( y = \sum_{i=1}^{m} \beta_i v_{\lambda_i} \). Let

\[
x = \sum_{i=1}^{m} \beta_i u_{\lambda_i}
\]

Then

\[
T(x) = T \left( \sum_{i=1}^{m} \beta_i u_{\lambda_i} \right)
\]

\[
= \sum_{i=1}^{m} \beta_i T(u_{\lambda_i}) \quad \text{(} T \text{ linear)}
\]

\[
= \sum_{i=1}^{m} \beta_i v_{\lambda_i}
\]

\[
= \sum_{i=1}^{m} \beta_i v_{\lambda_i} = y
\]

so \( T \) is onto, so \( T \) is an isomorphism and \( X, Y \) are isomorphic. \( \square \)
\[ f(x) = \begin{cases} \frac{1}{x} & x \in (0, \infty) \\ 0 & x = 0 \end{cases} \]

\[ \lim_{x \to 0} f(x) = \infty \]

\[ J(f) = \int_0^1 e^{-x} \frac{f(x)}{1 + f'(x)^2} \, dx \]

\[ J(f) - J(g) \leq \int_0^1 \left| e^{-x} \left( \frac{f(x) (1 + g'(x)^2)}{(1 + f'(x)^2)(1 + g'(x)^2)} - g(x) \right) \right| \, dx \]
\[ \leq \int_0^1 e^{-x} \left| f(x)(1 + g'(x^2)) - g(x)(1 + f'(x)^2) \right| \, dx \]
\[ \leq \int_0^1 e^{-x} \left| f(x) - g(x) + (f(x)g'(x)^2 - g(x)f'(x))^2 \right| \, dx \]
\[ p(x, A) = 0 \Rightarrow x \in \bar{A} \]

\[ p(x, A) = \inf_{a \in A} d(x, a) \]

\[ \forall n \exists a_n \in A \text{ s.t.} \]
\[ 0 \leq d(x, a_n) \leq \frac{1}{n} \]
\[ \Rightarrow a_n \to x \]
\[ \exists a_n \in A \Rightarrow x \in \bar{A} \]

\[ \bar{A} = A \cup \text{limit points of } A \]
\[ x \in \bar{A} \Rightarrow d(x, A) = 0 \]
$y \subseteq A \cup B$, $\forall x \in A \Rightarrow y \cap B = \emptyset$

$\neg y \in [x]$

$[x] \cap y \neq \emptyset$

$\neg y \notin [x]$

$U[x]$ vs. $[y]$

$\exists \varepsilon > 0$ s.t. $B_{\varepsilon}(y) \cap [x] = \emptyset$

$[x]$ open

$\neg \exists x \in O_x \wedge x \in O_x$ s.t. $O_x \subseteq [x]$

$\exists \varepsilon > 0$ s.t. $B_{\varepsilon}(x) \subseteq O_x$

$\forall z \in [x] \Rightarrow [z] = [x]$

$\exists x \in O_Z \wedge z \in O_Z$ s.t. $O_Z \subseteq [x] = [z]$
Suppose \( \exists x, y \text{ s.t. } y \notin [x] \)

\[
y = X = \bigcup_{x \in X} [x] = \left( \bigcup_{x \neq y} [x] \right) \cup [y]
\]

\[
[y] \cap \left( \bigcup_{x \notin [y]} [x] \right) = \emptyset
\]