Economics 204 Summer/Fall 2020
Lecture 4–Thursday July 30, 2020

Section 2.4. Open and Closed Sets

Definition 1 Let (X, d) be a metric space. A set $A \subseteq X$ is *open* if

$$\forall x \in A \ \exists \varepsilon > 0 \text{ s.t. } B_\varepsilon(x) \subseteq A$$

A set $C \subseteq X$ is *closed* if $X \setminus C$ is open.

See Figure 1.

Example: (a, b) is open in the metric space $\mathbf{E}^1$ ($\mathbf{R}$ with the usual Euclidean metric). Given $x \in (a, b)$, $a < x < b$. Let

$$\varepsilon = \min\{x - a, b - x\} > 0$$

Then

$$\begin{aligned} y \in B_\varepsilon(x) &\Rightarrow y \in (x - \varepsilon, x + \varepsilon) \\ &\subseteq (x - (x - a), x + (b - x)) \\ &= (a, b) \end{aligned}$$

so $B_\varepsilon(x) \subseteq (a, b)$, so (a, b) is open.

Notice that ε depends on x; in particular, ε gets smaller as x nears the boundary of the set.

Example: In $\mathbf{E}^1$, $[a, b]$ is closed. $\mathbf{R} \setminus [a, b] = (-\infty, a) \cup (b, \infty)$ is a union of two open sets, which must be open.

Example: In the metric space $[0, 1]$, $[0, 1]$ is open. With $[0, 1]$ as the underlying metric space, $B_\varepsilon(0) = \{x \in [0, 1] : |x - 0| < \varepsilon\} = [0, \varepsilon)$.

Thus, openness and closedness depend on the underlying metric space as well as on the set.

Example: Most sets are neither open nor closed. For example, in $\mathbf{E}^1$, $[0, 1] \cup (2, 3)$ is neither open nor closed.

Example: An open set may consist of a single point. For example, if $X = \mathbf{N}$ and $d(m, n) = |m - n|$, then $B_{1/2}(1) = \{m \in \mathbf{N} : |m - 1| < 1/2\} = \{1\}$. Since 1 is the only element of the set $\{1\}$ and $B_{1/2}(1) = \{1\} \subseteq \{1\}$, the set $\{1\}$ is open.

Example: In any metric space (X, d) both $\emptyset$ and X are open, and both $\emptyset$ and X are closed. To see that $\emptyset$ is open, note that the statement

$$\forall x \in \emptyset \ \exists \varepsilon > 0 \ B_\varepsilon(x) \subseteq \emptyset$$

is vacuously true since there aren't any $x \in \emptyset$. To see that X is open, note that since $B_\varepsilon(x)$ is by definition $\{z \in X : d(z,x) < \varepsilon\}$, it is trivially contained in X. Since $\emptyset$ is open, X is closed; since X is open, $\emptyset$ is closed.

Example: Open balls are open sets. Suppose $y \in B_\varepsilon(x)$. Then $d(x,y) < \varepsilon$. Let $\delta = \varepsilon - d(x,y) > 0$. If $d(z,y) < \delta$, then

$$\begin{aligned} d(z,x) &\leq d(z,y) + d(y,x) \\ &< \delta + d(x,y) \\ &= \varepsilon - d(x,y) + d(x,y) \\ &= \varepsilon \end{aligned}$$

so $B_\delta(y) \subseteq B_\epsilon(x)$, so $B_\varepsilon(x)$ is open.

Theorem 2 (Thm. 4.2) *Let (X,d) be a metric space. Then*

1. *$\emptyset$ and X are both open, and both closed.*
2. *The union of an arbitrary (finite, countable, or uncountable) collection of open sets is open.*
3. *The intersection of a finite collection of open sets is open.*

Proof:

1. We have already shown this.
2. Suppose $\{A_\lambda\}_{\lambda\in\Lambda}$ is a collection of open sets.

$$\begin{aligned} x \in \bigcup_{\lambda\in\Lambda} A_\lambda &\Rightarrow \exists \lambda_0 \in \Lambda \text{ s.t. } x \in A_{\lambda_0} \\ &\Rightarrow \exists \varepsilon > 0 \text{ s.t. } B_\varepsilon(x) \subseteq A_{\lambda_0} \subseteq \bigcup_{\lambda\in\Lambda} A_\lambda \end{aligned}$$

   so $\cup_{\lambda\in\Lambda} A_\lambda$ is open.

3. Suppose $A_1, \ldots, A_n \subseteq X$ are open sets. If $x \in \cap_{i=1}^n A_i$, then

$$x \in A_1, x \in A_2, \ldots, x \in A_n$$

   so

$$\exists \varepsilon_1 > 0, \ldots, \varepsilon_n > 0 \text{ s.t. } B_{\varepsilon_1}(x) \subseteq A_1, \ldots, B_{\varepsilon_n}(x) \subseteq A_n$$

   Let

$$\varepsilon = \min\{\varepsilon_1, \ldots, \varepsilon_n\} > 0$$

(Note this is where we need the fact that we are taking a finite intersection. The infimum of an infinite set of positive numbers could be zero. And the intersection of an infinite collection of open sets need not be open.)

Then

$$B_\varepsilon(x) \subseteq B_{\varepsilon_1}(x) \subseteq A_1, \ldots, B_\varepsilon(x) \subseteq B_{\varepsilon_n}(x) \subseteq A_n$$

so

$$B_\varepsilon(x) \subseteq \bigcap_{i=1}^{n} A_i$$

which proves that $\cap_{i=1}^{n} A_i$ is open.

■

Definition 3

- The *interior* of A, denoted $\operatorname{int} A$, is the largest open set contained in A (the union of all open sets contained in A).
- The *closure* of A, denoted $\bar{A}$, is the smallest closed set containing A (the intersection of all closed sets containing A)
- The *exterior* of A, denoted $\operatorname{ext} A$, is the largest open set contained in $X \setminus A$.
- The *boundary* of A, denoted $\partial A = \overline{(X \setminus A)} \cap \bar{A}$

Example: Let $A = [0,1] \cup (2,3)$. Then

$$\begin{aligned}
\operatorname{int} A &= (0,1) \cup (2,3) \\
\bar{A} &= [0,1] \cup [2,3] \\
\operatorname{ext} A &= \operatorname{int}(X \setminus A) \\
&= (-\infty, 0) \cup (1,2) \cup (3, +\infty) \\
\partial A &= \overline{(X \setminus A)} \cap \bar{A} \\
&= ((-\infty, 0] \cup [1,2] \cup [3, +\infty)) \cap ([0,1] \cup [2,3]) \\
&= \{0,1,2,3\}
\end{aligned}$$

Theorem 4 (Thm. 4.13) *A set A in a metric space (X,d) is closed if and only if*

$$\{x_n\} \subset A, x_n \to x \in X \Rightarrow x \in A$$

Proof:[1] Suppose A is closed. Then $X \setminus A$ is open. Consider a convergent sequence $x_n \to x \in X$, with $x_n \in A$ for all n. If $x \notin A$, $x \in X \setminus A$, so there is some $\varepsilon > 0$ such that $B_\varepsilon(x) \subseteq X \setminus A$. (See Figure 2.) Since $x_n \to x$, there exists $N(\varepsilon)$ such that

$$\begin{aligned}
n > N(\varepsilon) &\Rightarrow x_n \in B_\varepsilon(x) \\
&\Rightarrow x_n \in X \setminus A \\
&\Rightarrow x_n \notin A
\end{aligned}$$

[1] This is different from the proof in de la Fuente: he puts the meat of the proof into Theorem 4.12

contradiction. Therefore,

$$x_n \subset A, x_n \to x \in X \Rightarrow x \in A$$

Conversely, suppose

$$\{x_n\} \subset A, x_n \to x \in X \Rightarrow x \in A$$

We need to show that A is closed, i.e. $X \setminus A$ is open. Suppose not, so $X \setminus A$ is not open. Then there exists $x \in X \setminus A$ such that for every $\varepsilon > 0$,

$$B_\varepsilon(x) \not\subseteq X \setminus A$$

so there exists $y \in B_\varepsilon(x)$ such that $y \notin X \setminus A$. Then $y \in A$, hence

$$B_\varepsilon(x) \cap A \neq \emptyset$$

See Figure 3. Construct a sequence $\{x_n\}$ as follows: for each n, choose $x_n \in B_{\frac{1}{n}}(x) \cap A$. Given $\varepsilon > 0$, we can find $N(\varepsilon)$ such that $N(\varepsilon) > \frac{1}{\varepsilon}$ by the Archimedean Property, so $n > N(\varepsilon) \Rightarrow \frac{1}{n} < \frac{1}{N(\varepsilon)} < \varepsilon$, so $x_n \to x$. Then $\{x_n\} \subseteq A$, $x_n \to x$, so $x \in A$, contradiction. Therefore, $X \setminus A$ is open, so A is closed. ■

Section 2.5. Limits of Functions

Note: Read this section of de la Fuente on your own.

Note that we may have $\lim_{x \to a} f(x) = y$ even though

- f is not defined at a; or
- f is defined at a but $f(a) \neq y$.

The existence and value of the limit depends on values of f *near* a but not *at* a.

Section 2.6. Continuity in Metric Spaces

Definition 5 Let (X, d) and (Y, ρ) be metric spaces. A function $f : X \to Y$ is *continuous at a point* $x_0 \in X$ if $\forall \varepsilon > 0\ \exists \delta(x_0, \varepsilon) > 0$ s.t. $d(x, x_0) < \delta(x_0, \varepsilon) \Rightarrow \rho(f(x), f(x_0)) < \varepsilon$.

f is *continuous* if it is continuous at every element of its domain.

Note that δ depends on x_0 and ε.

This is a straightforward generalization of the definition of continuity in $\mathbf{R}$. Continuity at x_0 requires:

- $f(x_0)$ is defined; and

- either
  - x_0 is an isolated point of X, i.e. $\exists \varepsilon > 0$ s.t. $B_\varepsilon(x) = \{x\}$; or
  - $\lim_{x \to x_0} f(x)$ exists and equals $f(x_0)$

Suppose $f : X \to Y$ and $A \subseteq Y$. Define $f^{-1}(A) = \{x \in X : f(x) \in A\}$.

Theorem 6 (Thm. 6.14) *Let (X, d) and (Y, ρ) be metric spaces, and $f : X \to Y$. Then f is continuous if and only if*

$$f^{-1}(A) \textit{ is open in } X \; \forall A \subseteq Y \textit{ s.t. } A \textit{ is open in } Y$$

Proof:[2] Suppose f is continuous. Given $A \subseteq Y$, A open, we must show that $f^{-1}(A)$ is open in X. Suppose $x_0 \in f^{-1}(A)$. Let $y_0 = f(x_0) \in A$. Since A is open, we can find $\varepsilon > 0$ such that $B_\varepsilon(y_0) \subseteq A$. Since f is continuous, there exists $\delta > 0$ such that

$$\begin{aligned} d(x, x_0) < \delta \;&\Rightarrow\; \rho(f(x), f(x_0)) < \varepsilon \\ &\Rightarrow\; f(x) \in B_\varepsilon(y_0) \\ &\Rightarrow\; f(x) \in A \\ &\Rightarrow\; x \in f^{-1}(A) \end{aligned}$$

so $B_\delta(x_0) \subseteq f^{-1}(A)$, so $f^{-1}(A)$ is open. (See Figure 4.)

Conversely, suppose

$$f^{-1}(A) \text{ is open in } X \; \forall A \subseteq Y \text{ s.t. } A \text{ is open in } Y$$

We need to show that f is continuous. Let $x_0 \in X$, $\varepsilon > 0$. Let $A = B_\varepsilon(f(x_0))$. A is an open ball, hence an open set, so $f^{-1}(A)$ is open in X. $x_0 \in f^{-1}(A)$, so there exists $\delta > 0$ such that $B_\delta(x_0) \subseteq f^{-1}(A)$. (See Figure 5.)

$$\begin{aligned} d(x, x_0) < \delta \;&\Rightarrow\; x \in B_\delta(x_0) \\ &\Rightarrow\; x \in f^{-1}(A) \\ &\Rightarrow\; f(x) \in A \\ &\Rightarrow\; \rho(f(x), f(x_0)) < \varepsilon \end{aligned}$$

Thus, we have shown that f is continuous at x_0; since x_0 is an arbitrary point in X, f is continuous.■

Theorem 7 (Slightly weaker version of Thm. 6.10) *Let (X, d_X), (Y, d_Y) and (Z, d_Z) be metric spaces. If $f : X \to Y$ and $g : Y \to Z$ are continuous, then $g \circ f : X \to Z$ is continuous.*

[2] We give a direct proof; de la Fuente works via closed sets.

Proof: Suppose $A \subseteq Z$ is open. Since g is continuous, $g^{-1}(A)$ is open in Y; since f is continuous, $f^{-1}(g^{-1}(A))$ is open in X.

We claim that

$$f^{-1}(g^{-1}(A)) = (g \circ f)^{-1}(A)$$

Observe

$$\begin{aligned} x \in f^{-1}(g^{-1}(A)) &\Leftrightarrow f(x) \in g^{-1}(A) \\ &\Leftrightarrow g(f(x)) \in A \\ &\Leftrightarrow (g \circ f)(x) \in A \\ &\Leftrightarrow x \in (g \circ f)^{-1}(A) \end{aligned}$$

which establishes the claim. This shows that $(g \circ f)^{-1}(A)$ is open in X, so $g \circ f$ is continuous. ■

Definition 8 [Uniform Continuity] Suppose $f : (X, d) \to (Y, \rho)$. f is *uniformly continuous* if

$$\forall \varepsilon > 0 \ \exists \delta(\varepsilon) > 0 \text{ s.t. } \forall x_0 \in X, \ d(x, x_0) < \delta(\varepsilon) \Rightarrow \rho(f(x), f(x_0)) < \varepsilon$$

Notice the important contrast with continuity: f is continuous means

$$\forall x_0 \in X, \varepsilon > 0 \ \exists \delta(x_0, \varepsilon) > 0 \text{ s.t. } d(x, x_0) < \delta(x_0, \varepsilon) \Rightarrow \rho(f(x), f(x_0)) < \varepsilon$$

Example: Consider

$$f(x) = \frac{1}{x}, \ \ x \in (0, 1]$$

f is continuous (why?). We will show that f is **not** uniformly continuous. Fix $\varepsilon > 0$ and $x_0 \in (0, 1]$. If $x = \frac{x_0}{1+\varepsilon x_0}$, then

$$\begin{aligned} 1 + \varepsilon x_0 &> 1 \\ x = \frac{x_0}{1 + \varepsilon x_0} &< x_0 \\ \frac{1}{x} - \frac{1}{x_0} &> 0 \\ |f(x) - f(x_0)| &= \left| \frac{1}{x} - \frac{1}{x_0} \right| \\ &= \frac{1}{x} - \frac{1}{x_0} \\ &= \frac{1 + \varepsilon x_0}{x_0} - \frac{1}{x_0} \\ &= \frac{\varepsilon x_0}{x_0} \\ &= \varepsilon \end{aligned}$$

Thus, $\delta(x_0,\varepsilon)$ must be chosen small enough so that

$$\left|\frac{x_0}{1+\varepsilon x_0} - x_0\right| \geq \delta(x_0,\varepsilon)$$

$$\begin{aligned}
\delta(x_0,\varepsilon) &\leq x_0 - \frac{x_0}{1+\varepsilon x_0} \\
&= \frac{\varepsilon(x_0)^2}{1+\varepsilon x_0} \\
&< \varepsilon(x_0)^2
\end{aligned}$$

which converges to zero as $x_0 \to 0$. (See Figure 6.) So there is no $\delta(\varepsilon)$ that will work for all $x_0 \in (0,1]$.

Example: If $f : \mathbf{R} \to \mathbf{R}$ and $f'(x)$ is defined and uniformly bounded on an interval $[a,b]$, then $f(x)$ is uniformly continuous on $[a,b]$. However, even a function with an unbounded derivative may be uniformly continuous. Consider

$$f(x) = \sqrt{x}, \;\; x \in [0,1]$$

f is continuous (why?). We will show that f is uniformly continuous. Given $\varepsilon > 0$, let $\delta = \varepsilon^2$. Then given any $x_0 \in [0,1]$, $|x - x_0| < \delta$ implies by the Fundamental Theorem of Calculus

$$\begin{aligned}
|f(x) - f(x_0)| &= \left|\int_{x_0}^{x} \frac{1}{2\sqrt{t}}\,dt\right| \\
&\leq \int_0^{|x-x_0|} \frac{1}{2\sqrt{t}}\,dt \\
&= \sqrt{|x-x_0|} \\
&< \sqrt{\delta} \\
&= \sqrt{\varepsilon^2} \\
&= \varepsilon
\end{aligned}$$

Thus, f is uniformly continuous on $[0,1]$, even though $f'(x) \to \infty$ as $x \to 0$.

Definition 9 Let X, Y be normed vector spaces, $E \subseteq X$. $f : X \to Y$ is *Lipschitz on E* if

$$\exists K > 0 \text{ s.t. } \|f(x) - f(z)\|_Y \leq K\|x - z\|_X \;\; \forall x, z \in E$$

f is *locally Lipschitz* on E if

$$\forall x_0 \in E \; \exists \varepsilon > 0 \text{ s.t. } f \text{ is Lipschitz on } B_\varepsilon(x_0) \cap E$$

Remark: de la Fuente only defines Lipschitz and locally Lipschitz in the context of normed vector spaces. The notions can also be defined analogously in metric spaces as follows: Let (X, d) and (Y, ρ) be metric spaces, $E \subseteq X$. $f : X \to Y$ is *Lipschitz on* E if

$$\exists K > 0 \text{ s.t. } \rho(f(x), f(z)) \leq K d(x, z) \ \ \forall x, z \in E$$

Similarly, f is *locally Lipschitz* on E if

$$\forall x_0 \in E \ \exists \varepsilon > 0 \text{ s.t. } f \text{ is Lipschitz on } B_\varepsilon(x_0) \cap E$$

Lipschitz continuity is stronger than either continuity or uniform continuity:

$$\begin{aligned} \text{locally Lipschitz} &\Rightarrow \text{continuous} \\ \text{Lipschitz} &\Rightarrow \text{uniformly continuous} \end{aligned}$$

Every C^1 function is locally Lipschitz. (Recall that a function $f : \mathbf{R}^m \to \mathbf{R}^n$ is said to be C^1 if all its first partial derivatives exist and are continuous.)

Definition 10 [3] Let (X, d) and (Y, ρ) be metric spaces. A function $f : X \to Y$ is called a *homeomorphism* if it is one-to-one, onto, continuous, and its inverse function is continuous.

Now suppose that f is a homeomorphism and $U \subset X$. Let $g : Y \to X$ be the inverse of f, so $g \circ f : X \to X$ is the identity on X, and $f \circ g : Y \to Y$ is the identity on Y.

$$\begin{aligned} y \in g^{-1}(U) &\Leftrightarrow g(y) = f^{-1}(y) \in U \\ &\Leftrightarrow y \in f(U) \\ U \text{ open in } X &\Rightarrow g^{-1}(U) \text{ is open in } (f(X), \rho) \\ &\Rightarrow f(U) \text{ is open in } (f(X), \rho) \end{aligned}$$

This says that (X, d) and $\left(f(X), \rho|_{f(X)}\right)$ are identical in terms of properties that can be characterized solely in terms of open sets; such properties are called "topological properties."

[3] This is the standard definition; de la Fuente instead omits the requirement that f be onto, and requires that f^{-1} be continous on $f(X)$. See the Corrections handout for a correction to Theorem 6.21

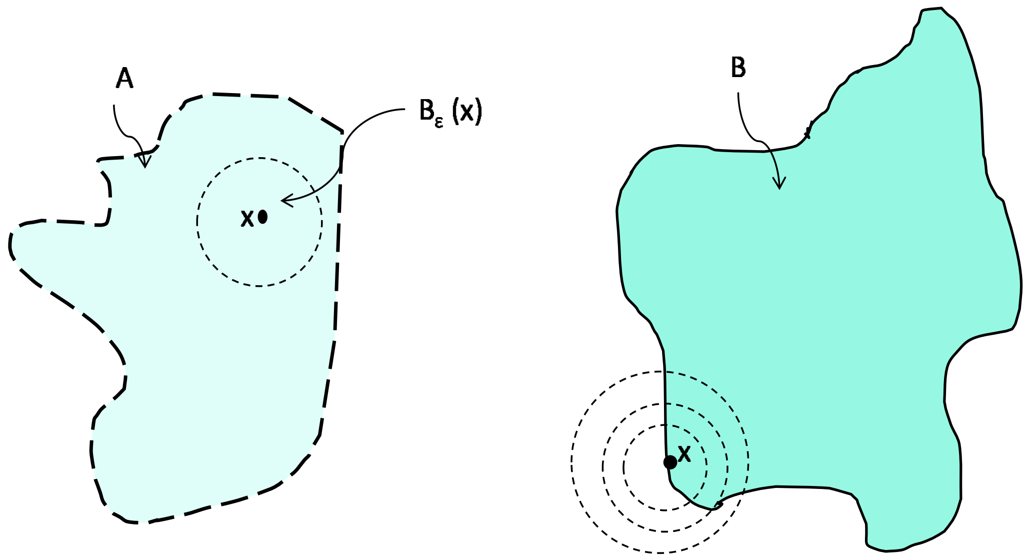

Figure 1: A is open: for every $x \in A$ there is some $\varepsilon > 0$ such that $B_\varepsilon(x) \subseteq A$. B is not open: for x depicted in the picture $\nexists \varepsilon > 0$ such that $B_\varepsilon(x) \subseteq B$.

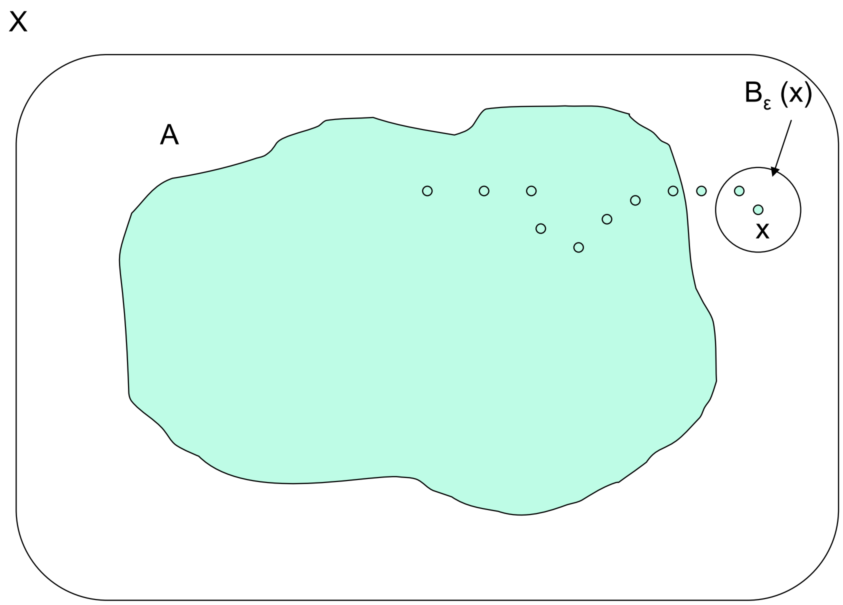

Figure 2: Sequences and closed sets

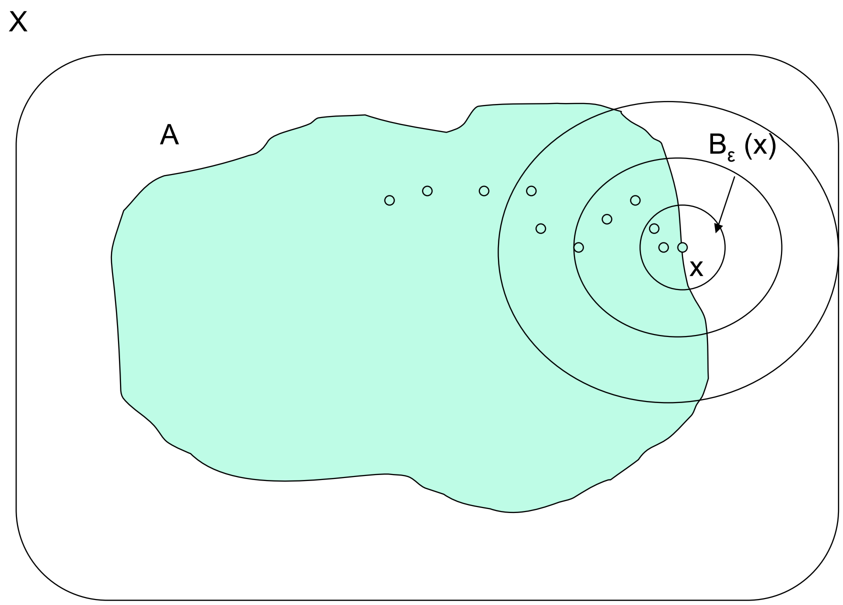

Figure 3: Sequences and closed sets

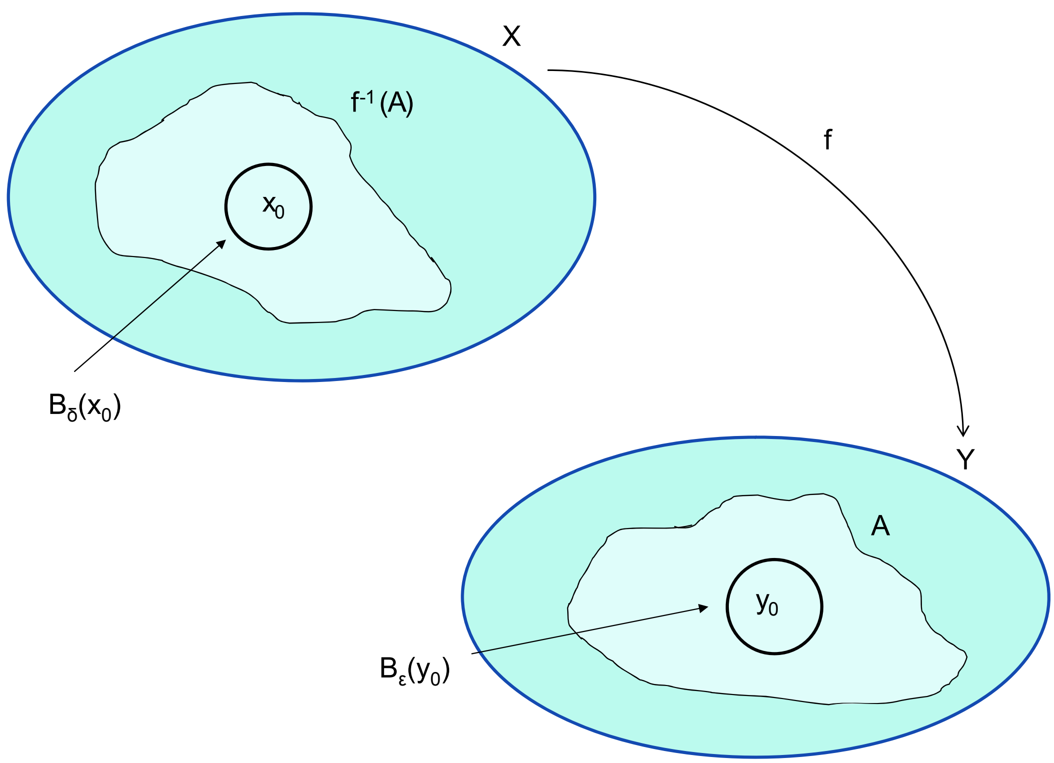

Figure 4: Proof of Theorem 6.

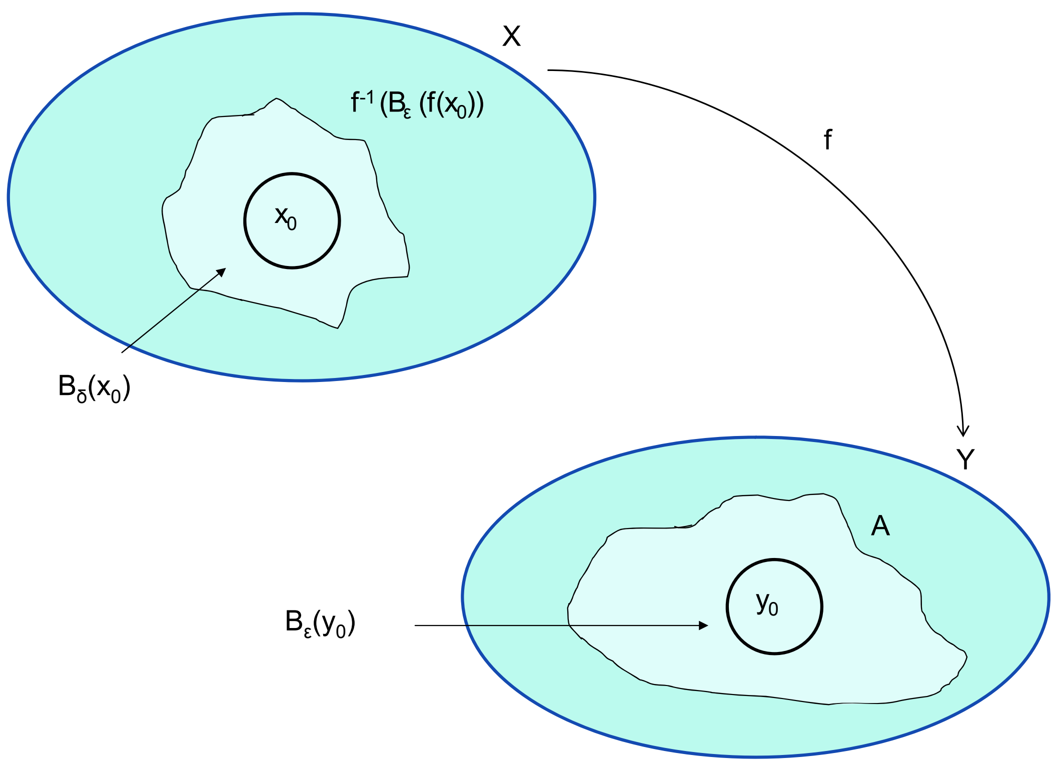

Figure 5: Proof of Theorem 6.

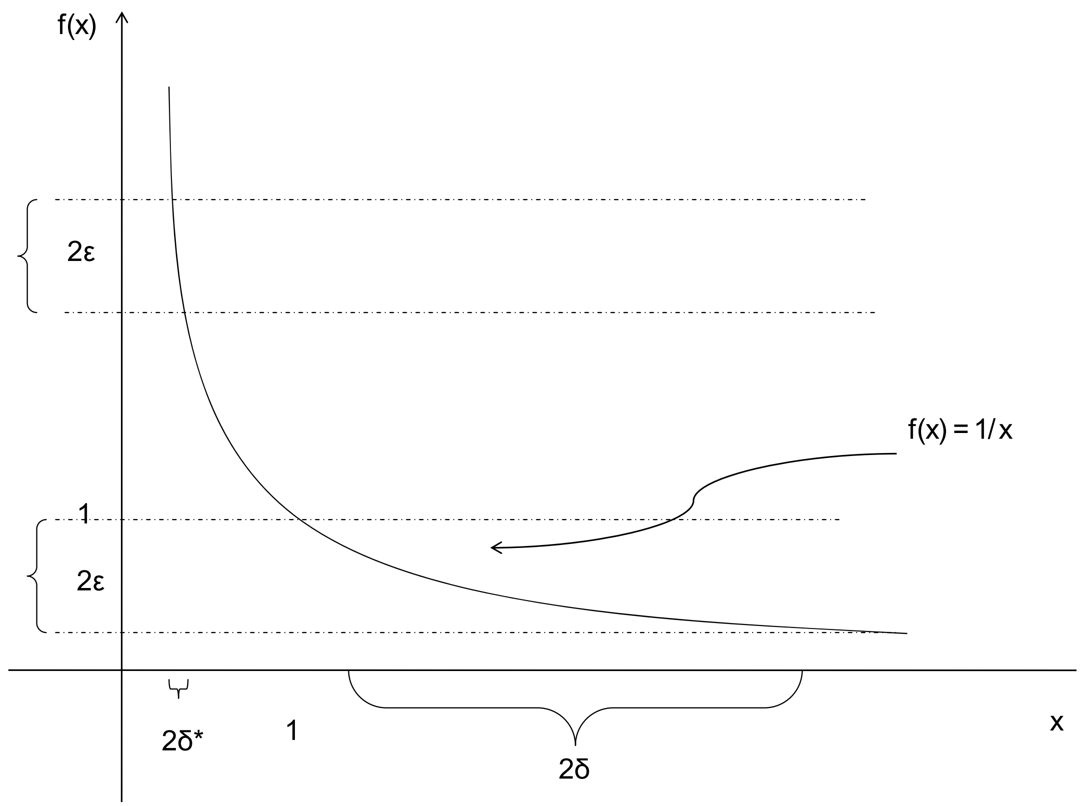

Figure 6: $f(x) = \frac{1}{x}$ is not uniformly continuous.