Econ 204 – Problem Set 2

Due Tuesday, August 4

1. Give an example of a complete metric space which is homeomorphic to an incomplete metric space.

2. Let (E, d) be a metric space and $S \subset E$ a subset. Show that $A \subset S$ is open relative to S if and only if $A = S \cap U$ for some $U \subset E$ open.[1]

3. Let (X, d) be a metric space. Assume $f : X \to \mathbb{R}$ and $g : X \to \mathbb{R}$ are uniformly continuous on (X, d) and $(\mathbb{R}, |\cdot|)$, with $|\cdot|$ the absolute-value norm.

   (a) Show that $f + g : X \to \mathbb{R}$ is uniformly continuous, where $(f + g)(x) = f(x) + g(x)$.

   (b) Show that $\max\{f, g\} : X \to \mathbb{R}$ is uniformly continuous, where $\max\{f, g\}(x) = \max\{f(x), g(x)\}$.

   (c) Give a counterexample to the following statement: $f \cdot g : X \to \mathbb{R}$ *is uniformly continuous on* (X, d) *and* $(\mathbb{R}, |\cdot|)$, where $f \cdot g = f(x) \cdot g(x)$.

4. A function $f : X \to Y$ is open if $\forall A \subset X$ such that A is open, $f(A)$ is open. Show that any continuous open function from $\mathbb{R}$ into $\mathbb{R}$ is strictly monotonic.

5. Prove that a metric space (X, d) is discrete if and only if every function on X into any other metric space (Y, ρ), where Y has at least two distinct elements, is continuous.[2]

6. Suppose T is an operator on a complete metric space (X, d). Prove that the condition

$$d\left(T(x), T(y)\right) < d(x, y) \ \ \forall x, y \in X \ (x \neq y)$$

   does not guarantee the existence of a fixed point of T.

[1] $A \subset S$ is open relative to S if $\forall x \in A\ \exists r_x > 0$ such that $B_{r_x}(x) \cap S \subset A$.

[2] A metric space (X, d) is *discrete* if every subset $A \subset X$ is open. Notice that any set equipped with the discrete metric forms a discrete metric space, but not every discrete metric space necessarily has the discrete metric.