1. Give an example of a complete metric space which is homeomorphic to an incomplete metric space.

2. Let \((E, d)\) be a metric space and \(S \subseteq E\) a subset. Show that \(A \subseteq S\) is open relative to \(S\) if and only if \(A = S \cap U\) for some \(U \subseteq E\) open.\(^1\)

3. Let \((X, d)\) be a metric space. Assume \(f : X \to \mathbb{R}\) and \(g : X \to \mathbb{R}\) are uniformly continuous on \((X, d)\) and \((\mathbb{R}, |\cdot|)\), with \(|\cdot|\) the absolute-value norm.

   (a) Show that \(f + g : X \to \mathbb{R}\) is uniformly continuous, where \((f + g)(x) = f(x) + g(x)\).

   (b) Show that \(\max\{f, g\} : X \to \mathbb{R}\) is uniformly continuous, where \(\max\{f, g\}(x) = \max\{f(x), g(x)\}\).

   (c) Give a counterexample to the following statement: \(f \cdot g : X \to \mathbb{R}\) is uniformly continuous on \((X, d)\) and \((\mathbb{R}, |\cdot|)\), where \(f \cdot g = f(x) \cdot g(x)\).

4. A function \(f : X \to Y\) is open if \(\forall A \subseteq X\) such that \(A\) is open, \(f(A)\) is open. Show that any continuous open function from \(\mathbb{R}\) into \(\mathbb{R}\) is strictly monotonic.

5. Prove that a metric space \((X, d)\) is discrete if and only if every function on \(X\) into any other metric space \((Y, \rho)\), where \(Y\) has at least two distinct elements, is continuous.\(^2\)

6. Suppose \(T\) is an operator on a complete metric space \((X, d)\). Prove that the condition

\[
d(T(x), T(y)) < d(x, y) \quad \forall x, y \in X \ (x \neq y)
\]

does not guarantee the existence of a fixed point of \(T\).

\(^1\)A \(\subseteq S\) is open relative to \(S\) if \(\forall x \in A \exists r_x > 0\) such that \(B_{r_x}(x) \cap S \subseteq A\).

\(^2\)A metric space \((X, d)\) is discrete if every subset \(A \subseteq X\) is open. Notice that any set equipped with the discrete metric forms a discrete metric space, but not every discrete metric space necessarily has the discrete metric.