1. Let \((X, d)\) be a metric space:

(a) Let \(y \in X\) be given. Define the function \(d_y : X \to \mathbb{R}\) by

\[
d_y(x) = d(x, y)
\]

(1)

Show that \(d_y\) is a continuous function on \(X\) for each \(y \in X\).

(b) Let \(A\) be a subset of \(X\) and \(x \in X\). Recall that the distance from the point \(x\) to the set \(A\) is defined as:

\[
\rho(x, A) = \inf \{d(x, a) : a \in A\}
\]

(2)

Show that the closure of set \(A\) is the set of all points with zero distance to \(A\), that is:

\[
\overline{A} = \{x \in X : \rho(x, A) = 0\}
\]

(3)

(c) Now let \(A \subset X\) be a compact subset. Show that \(\rho(x, A) = d(x, a)\) for some \(a \in A\).

2. Let \(D\) be the space of all functions \(f : [0, 1] \to \mathbb{R}\) such that \(f\) is continuous and such that for some \(\varepsilon > 0\), \(f : (-\varepsilon, 1 + \varepsilon) \to \mathbb{R}\) is differentiable and \(f' : (-\varepsilon, 1 + \varepsilon) \to \mathbb{R}\) is continuous. For each \(f \in D\), let

\[
\|f\|_\infty = \sup \{|f(t)| : t \in [0, 1]\} \quad \text{and} \quad \|f'\|_\infty = \sup \{|f'(t)| : t \in [0, 1]\}
\]

(4)

Define the function \(\|\cdot\| : D \to \mathbb{R}_+\) by

\[
\|f\| := \|f\|_\infty + \|f'\|_\infty
\]

(5)

(a) Show that \((D, \|\cdot\|)\) is a normed vector space.

(b) Define the function \(J : D \to \mathbb{R}\) as

\[
J(f) = \int_0^1 e^{-x} \frac{f(x)}{1 + f'(x)^2} dx.
\]

(6)

Prove that \(J\) is continuous.

3. Let \(X\) be a connected space. Let \(\mathcal{R}\) be an equivalence relation on \(X\) such that for each \(x \in X\), there exists an open set \(\mathcal{O}_x\) containing \(x\) such that \(\mathcal{O}_x \subset [x]\). Show that \(\mathcal{R}\) has only one (distinct) equivalence class.

\[\text{In case of any problems with the exercises please email farzad@berkeley.edu}\]
4. Define the correspondence \( \Gamma : [0, 1] \to 2^{[0,1]} \) by:

\[
\Gamma(x) = \begin{cases} 
[0, 1] \cap \mathbb{Q} & \text{if } x \in [0, 1] \setminus \mathbb{Q}, \\
[0, 1] \setminus \mathbb{Q} & \text{if } x \in [0, 1] \cap \mathbb{Q}.
\end{cases}
\] 

(7)

Show that \( \Gamma \) is not continuous, but it is lower-hemicontinuous. Is \( \Gamma \) upper-hemicontinuous at any rational? At any irrational? Does this correspondence have a closed graph?

5. Let \( X \) be a metric space, and \( I : X \to \mathbb{R}_+ \) be a lower semi-continuous function. \(^2\)

(a) Prove that for every given \( \varepsilon > 0 \) there exists an open set \( U_\varepsilon \) containing \( x \in X \) such that

\[
\inf \{ I(y) : y \in U_\varepsilon \} \geq I(x) - \varepsilon.
\]

(8)

(b) Let \( x \in X \). For each \( n \in \mathbb{N} \) let

\[
m_n = \inf \{ I(y) : y \in B_{1/n}(x) \}.
\]

(9)

Show that \( \{m_n\} \) is an increasing sequence and that \( m_n \to I(x) \).

6. Let \( x \) and \( y \) be moving objects in \( \mathbb{R} \). Time is discrete, namely \( t \in \mathbb{Z}_+ := \{0\} \cup \mathbb{N} \). In addition, \( \beta > 1 \) is a fixed parameter. For \( a, b \in \mathbb{R} \), let \( \rho(a, b) := |a - b| \wedge 1 \) (as mentioned in the section, the symbol \( \wedge \) is sometimes used to refer to the minimum of two elements). Then for any \( x, y \in \mathbb{R}^\omega \), let

\[
d(x, y) = \sum_{t \in \mathbb{Z}_+} \beta^{-t} \rho(x_t, y_t)
\]

(10)

denotes the distance between \( x = (x_0, x_1, \ldots) \) and \( y = (y_0, y_1, \ldots) \), where \( x_t \) is the position of \( x \) at time \( t \) on the real line.

(a) Show that \( d \) is a metric on \( \mathbb{R}^\omega \).

(b) Show that \( (\mathbb{R}^\omega, d) \) is a bounded metric space.

(c) Is \( [0, 1]^\omega \) an open or closed subset of \( \mathbb{R}^\omega \)? (in either case present a proof)

(d) Is \( (\mathbb{R}^\omega, d) \) a complete metric space? (prove if yes, otherwise provide a counterexample)

(e) Is \( [0, 1]^\omega \) a totally bounded subset under \( d? \) Is it a compact subset?

\(^2\)A function \( I : X \to \mathbb{R} \) is called lower semi-continuous iff for every \( \alpha \) the set \( \{ x : I(x) > \alpha \} \) is open in \( X \).

\(^3\)For the definition of \( \mathbb{R}^\omega \) please refer to Q2 of ps.1