Econ 204 – Problem Set 4

Due Tuesday, August 11

1. Let A be an $n \times n$ matrix.
   (a) Show that if λ is an eigenvalue of A, then λ^k is an eigenvalue of A^k for $k \in \mathbb{N}$.
   (b) Show that if λ is an eigenvalue of the matrix A and A is invertible, then $1/\lambda$ is an eigenvalue of A^{-1}.
   (c) Find an expression for $\det(A)$ in terms of the eigenvalues of A.
   (d) The *eigenspace* of an eigenvalue λ_i is the kernel of $A - \lambda_i I$. Show that the eigenspace of any matrix A belonging to an eigenvalue λ_i is a vector space.
2. Let V be an n-dimensional vector space. Call a linear operator $T : V \to V$ *idempotent* if $T \circ T = T$. Prove that all such operators are diagonalizable (that is, any matrix representation $A = Mtx_U(T)$ is diagonalizable). What are the eigenvalues?
3. Let V be a finite-dimensional vector space and $W \subset V$ be a vector subspace. Prove that W has a complement in V, i.e., there exists a vector subspace $W' \subset V$ such that $W \cap W' = \{0\}$ and $W + W' = V$.
4. Let U and V be vector spaces. Suppose $T : U \to V$ is a linear transformation and $v \in V$. Prove that, if the preimage $T^{-1}(v)$ is non-empty, and $u \in T^{-1}(v)$, then $T^{-1}(v) = \{u + z | z \in \ker T\} = u + \ker T$.
5. Let V be a finite dimensional vector space and $T, S \in L(V, V)$. Prove that TS is invertible if and only if T and S are invertible.
6. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be given by $T(x, y) = (4x - 2y, x + y)$. Let V be the standard basis and $W = \{(5, 3), (1, 1)\}$ be another basis of $\mathbb{R}^2$.
   (a) Find $Mtx_V(T)$.
   (b) Find $Mtx_W(T)$.
   (c) Compute $T(4, 3)$ using the matrix representation of W.