Econ 204 – Problem Set 5[1]

Due Friday August 14, 2020

1. Let $f_n : \mathbb{R} \to \mathbb{R}$ be differentiable for each $n \in \mathbb{N}$ with $\left|f_n'(x)\right| \leq 1$ for all n and x. Assume,
$$\lim_{n\to\infty} f_n(x) = g(x) \tag{1}$$
for all x. Prove that $g : \mathbb{R} \to \mathbb{R}$ is Lipschitz-continuous.

2. Let $f : \mathbb{R} \to \mathbb{R}$ be a C^2 (twice continuously differentiable) function. The function and its second derivative are bounded, namely there exist $M, N > 0$ such that $\sup_{x\in\mathbb{R}} \left|f(x)\right| \leq M$ and $\sup_{x\in\mathbb{R}} \left|f''(x)\right| \leq N$. Show that $\sup_{x\in\mathbb{R}} \left|f'(x)\right| \leq 2\sqrt{MN}$.

3. The oscillation of an arbitrary function $f : [a, b] \to \mathbb{R}$ at $x \in [a, b]$ is [2]
$$\text{osc}_x f := \lim_{r\downarrow 0} \text{diam}\left(f\left([x - r, x + r]\right)\right), \tag{2}$$
where for every $x_1, x_2 \in [a, b]$, $f([x_1, x_2]) := \{y : y = f(x) \text{ for some } x \in [x_1, x_2]\}$. For $k > 0$, let D_k be the set of points with oscillation greater than or equal to k, i.e $D_k := \{x \in [a, b] : \text{osc}_x f \geq k\}$. Prove that D_k is closed.[3]

4. The goal of this exercise is to verify the **Banach-Steinhaus** theorem. Let $\{T_n\}$ be a sequence of bounded linear functions $T_n : X \to Y$ from a Banach (complete normed vector) space X into a normed vector space Y, such that $\{T_n(x)\}$ is bounded for every $x \in X$, that is for all $x \in X$ there exists $c_x \in \mathbb{R}_+$ such that:
$$\left\|T_n(x)\right\| \leq c_x \quad \forall n \in \mathbb{N} \tag{3}$$
Then, we want to show that the sequence of norms $\left\{\|T_n\|\right\}$ is bounded, that is there exists $c > 0$ such that $\|T_n\| \leq c$ for all $n \in \mathbb{N}$.

   (a) For every $k \in \mathbb{N}$ let $A_k \subseteq X$ be the set of all $x \in X$ such that $\left\|T_n(x)\right\| \leq k$ for all n. Show that A_k is closed under the X-norm.

   (b) Use equation (3) to show that $X = \bigcup_{k\in\mathbb{N}} A_k$.

   (c) The **Baire's** theorem states that in this case since X is complete, there exists some A_{k_0} that contains an open ball, say $B_\varepsilon(x_0) \subseteq A_{k_0}$. Take this result as given, and prove there exists some constant $c > 0$ such that
$$\|T_n\| \leq c \quad \forall n \in \mathbb{N}. \tag{4}$$

[1] In case of any problems with the exercises please email farzad@berkeley.edu
[2] The symbol '↓' means that r *decreases* to 0 along the limit.
[3] This question is part of the exercise 19 in chapter 3 of the second edition of *Real Mathematical Analysis*, Charles Chapman Pugh.

Hint: For every nonzero $x \in X$ there exists $\gamma > 0$ such that $x = \frac{1}{\gamma}(z - x_0)$, where $x_0, z \in B_\varepsilon(x_0)$ and $\gamma > 0$.

5. Suppose $\Psi : X \to 2^X$ is a non-empty and compact-valued upper-hemicontinuous correspondence. The metric space X is compact. Show that there exists a non-empty compact set $C \subset X$ such that $\Psi(C) = C$ (you can use the exercises that are proved in the sections).