Econ 204 – Problem Set 6

Due Monday, August 17

1. Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a C^1 function. Define $F : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ by

$$F(x, \omega) = f(x) + \omega$$

   Show that there is a set $\Omega_0 \subset \mathbb{R}^n$ of Lebesgue measure zero such that, if $\omega \notin \Omega_0$, then for each x_0 satisfying $F(x_0, \omega_0) = 0$ there is an open set U containing x_0, an open set V containing ω_0, and a C^1 function $h : V \to U$ such that for all $\omega \in V$, $x = h(\omega)$ is the unique element of U satisfying $F(x, \omega) = 0$.

2. Let $A \in \mathbb{R}^{m\times n}$ and $b \in \mathbb{R}^{m\times 1}$. Then, show that exactly one of the following two conditions hold:

   - $\exists x \in \mathbb{R}^n$ such that $Ax = b$, with $x \geq 0$;
   - $\exists y \in \mathbb{R}^{1\times m}$ such that $A'y \geq 0$, and $y'b < 0$.

   Hint: you may want to use the following definition and its properties. If $v_1, v_2, ..., v_n$ are the columns of A, define

$$Q = \text{cone}(A) \equiv \left\{ s \in \mathbb{R}^m : s = \sum_{i=1}^{n} \lambda_i v_i, \lambda_i \geq 0, \forall i \right\},$$

   i.e., Q is the set of all conic combinations of the columns of A. Note that Q is non-empty ($0 \in Q$), and assume it is closed and convex (you should be able to prove this!).

3. Call a vector $\pi \in \mathbb{R}^n$ a *probability vector* if

$$\sum_{i}^{n} \pi_i = 1 \text{ and } \pi_i \geq 0 \ \forall i$$

   We say there are n states of the world, and π_i is the probability that state i occurs. Suppose there are two traders (trader 1 and trader 2) who each have a set of prior probability distributions (Π_1 and Π_2) which are nonempty, convex, and compact. Call a *trade* a vector $f \in \mathbb{R}^n$, which denotes the net transfer trader 1 receives in each state of the world (and thus $-f$ is the net transfer trader 2 receives in each state of the world). A trade is *agreeable* if

$$\inf_{\pi\in\Pi_1} \sum_{i=1}^{n} \pi_i f_i > 0 \ \text{ and } \ \inf_{\pi\in\Pi_2} \sum_{i=1}^{n} \pi_i(-f_i) > 0$$

   Prove that there exists an agreeable trade if and only if there is no common prior (that is, $\Pi_1 \cap \Pi_2 = \varnothing$).

4. (a) Let $A \subset \mathbb{R}^n$ be a convex set, and $\lambda_1, \lambda_2, ..., \lambda_p \geq 0$, with $\sum_{i=1}^{p} \lambda_i = 1$. Prove that, if $x_1, x_2, ..., x_p \in A$, then $\sum_{i=1}^{p} \lambda_i x_i \in A$.

   (b) The sum $\sum_{i=1}^{p} \lambda_i x_i$ defined in (a) is called a convex combination. The convex hull of a set S, denoted by $co(S)$, is the intersection of all convex sets which contain S. Prove that the set of all convex combinations of the elements of S is exactly $co(S)$.

5. a) *Berge's Maximum Theorem:* Let $X \subset \mathbb{R}^n$ and $Y \subset \mathbb{R}^m$. Consider the function $f : X \times Y \to \mathbb{R}$ and the correspondence $\Gamma : Y \twoheadrightarrow X$. Define $v(y) = \max_{x \in \Gamma(y)} f(x, y)$ and $\Omega(y) = \arg\max_{x \in \Gamma(y)} f(x, y)$. Suppose f and Γ are continuous, and that Γ has non-empty compact values. Show that v is continuous and Ω is uhc with non-empty compact values. *Hint*: you may find useful to use the sequential definitions of uhc and lhc.

   b) Assume that Γ also has convex values. Show that if f is quasi-concave in x, Ω has convex values.[1]

   c) Let $\mathcal{S}(I, (u^i, S^i, \Gamma^i)_{i \in I})$ denote a *social game*, where I is the (finite) set of players, and $u^i : \prod_{j \in I} S^j \to \mathbb{R}$ is the objective function of player $i \in I$ defined over $s = \left(s^j; j \in I\right) \in \prod_{j \in I} S^j$, with $S^j \subset \mathbb{R}^{n_j}$, $n_j > 0$. Each player i chooses $s^i \in \arg\max_{s \in \Gamma^i(s_{-i})} u^i(s, s_{-i})$, with $s_{-i} := (s_j; j \in I \setminus \{i\})$, and $\Gamma^i(s_{-i}) \subset S^i$. Define an equilibrium for the social game $\mathcal{S}(I, (u^i, S^i, \Gamma^i)_{i \in I})$ as a vector $\bar{s} = \left(\bar{s}^i; i \in I\right)$ such that, $\forall i \in I$, $u^i(\bar{s}) \geq u^i(s, \bar{s}_{-i})$, $\forall s \in \Gamma^i\left(\bar{s}_{-i}\right)$, where $\bar{s}_{-i} := \left(\bar{s}^j; j \neq i\right)$.

      Assume S^i is convex, compact, and non-empty for each $i \in I$, and that u^i is continuous and quasi-concave in s^i for each $i \in I$. Use the previous parts of this question to show that, if $\{\Gamma^i\}_{i \in I}$ are continuous and have compact, convex, and non-empty values, then an equilibrium for $\mathcal{S}(I, (u^i, S^i, \Gamma^i)_{i \in I})$ exists.

6. Solve the following differential equation: $y'' - 5y' + 4y = e^{4x}$. Concretely, provide (i) the general solution of the homogeneous differential equation, and (ii) the particular and general solutions of the inhomogeneous differential equation. Solve explicitly for the constants using the following initial conditions: $y(0) = 3$, $y(0)' = \frac{19}{3}$.

[1] A function $f : X \to \mathbb{R}$ is quasi-concave if for all $x_1, x_2 \in X$ and $\lambda \in [0, 1]$, $f(\lambda x_1 + (1 - \lambda) x_2) \leq \max\{f(x_1), f(x_2)\}$.