Economics 204 Summer/Fall 2020
Final Exam – Suggested Solutions

Answer all of the questions below. Be as complete, correct, and concise as possible. There are 7 questions for a total of 180 points possible; point values for each problem are in parentheses. For questions with subparts, each subpart is worth the same number of points. Use the points as a guide to allocating your time.

1. (15) Let A and B be $n \times n$ matrices that are similar, so there exists an invertible $n \times n$ matrix P such that $A = P^{-1}BP$. Show that for every $k \in \mathbb{N}$, $A^k = P^{-1}B^kP$ (where M^k is the product of k copies of the $n \times n$ matrix M).

   (**Hint:** use induction.)

   Solution: For the base case $k = 1$, the claim follows by definition: $A = P^{-1}BP$. For the induction hypothesis, assume that the claim is true for some $k \geq 1$, so $A^k = P^{-1}B^kP$. Then for $k+1$,

$$A^{k+1} = A^k A = (P^{-1}B^kP)(P^{-1}BP)$$

   using the induction hypothesis and similarity of A and B. Thus

$$\begin{aligned} A^{k+1} = A^k A &= (P^{-1}B^kP)(P^{-1}BP) \\ &= P^{-1}B^k(PP^{-1})BP \\ &= P^{-1}B^kIBP \\ &= P^{-1}B^kBP \\ &= P^{-1}B^{k+1}P \end{aligned}$$

   (Here I is the $n \times n$ identity matrix.) Thus the claim is true for $k+1$. Thus by induction, $A^k = P^{-1}B^kP$ for every $k \in \mathbb{N}$.

2. (15) Let (X, d) and (Y, ρ) be metric spaces and $f, g : X \to Y$ be continuous functions. Let $E \subseteq X$ be a dense subset of X, that is, a set such that $\overline{E} = X$. Show that if $f(z) = g(z)$ for all $z \in E$, then $f = g$, that is, $f(x) = g(x)$ for all $x \in X$.

   Solution: Let $x \in X$. Since $\overline{E} = X$, there is a sequence $\{x_n\} \subseteq E$ such that $x_n \to x$. Since f and g are continuous at x, $f(x_n) \to f(x)$ and $g(x_n) \to g(x)$. Since $x_n \in E$ for every n, $f(x_n) = g(x_n)$ for every n. Thus

$$f(x) = \lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} g(x_n) = g(x)$$

   Thus $f(x) = g(x)$. Since $x \in X$ was arbitrary, $f(x) = g(x)$ for all $x \in X$.

3. (30) Let X and Y be vector spaces over the same field F, and let $T : X \to Y$ be a linear transformation. Suppose $W \subseteq X$ is a subset of X that spans X, and $T(W) \subseteq Y$ is linearly independent.

(a.) Show that T is one-to-one.

Solution: Suppose $x \in X$ and $T(x) = 0$. Since W spans X, there exist $\alpha_1, \ldots, \alpha_n \in F$ and $w_1, \ldots, w_n \in W$ such that $x = \sum_{i=1}^{n} \alpha_i w_i$. Then

$$\begin{aligned} 0 = T(x) &= T(\sum_{i=1}^{n} \alpha_i w_i) \\ &= \sum_{i=1}^{n} \alpha_i T(w_i) \quad \text{(using linearity of } T) \end{aligned}$$

But $T(W)$ is a linearly independent subset of Y, so this implies $\alpha_i = 0$ for each $i = 1, \ldots, n$. So $x = \sum_{i=1}^{n} \alpha_i w_i = 0$. Thus $\ker T = \{0\}$. Since T is a linear transformation, this implies T is one-to-one.

(b.) Show that W is a basis for X.

Solution: Since W spans X by assumption, it suffices to show that W is linearly independent. To that end, suppose $\sum_{i=1}^{n} \alpha_i w_i = 0$ for some $\alpha_1, \ldots, \alpha_n \in F$ and $w_1, \ldots, w_n \in W$. Then since T is a linear transformation, $T(0) = 0$. Thus

$$0 = T(0) = T(\sum_{i=1}^{n} \alpha_i w_i) = \sum_{i=1}^{n} \alpha_i T(w_i)$$

using the linearity of T. But $T(W)$ is a linearly independent subset of Y, so this implies $\alpha_i = 0$ for each $i = 1, \ldots, n$. Thus W is linearly independent. Since W spans X by assumption, W is a basis for X.

4. (30) Let $f, g : \mathbb{R} \to \mathbb{R}$ be differentiable functions. Suppose that $f(0) = g(0)$ and $f'(x) \leq g'(x)$ for all $x \in \mathbb{R}$. Show that $f(x) \leq g(x)$ for all $x \geq 0$.

Solution: Let $h : \mathbb{R} \to \mathbb{R}$ be given by

$$h(x) = g(x) - f(x)$$

Since f and g are differentiable on $\mathbb{R}$ by assumption, h is differentiable on $\mathbb{R}$, and thus also continuous on $\mathbb{R}$. Also $h(0) = g(0) - f(0) = 0$, and $h'(x) = g'(x) - f'(x) \geq 0$ for all $x \in \mathbb{R}$.

Then let $x > 0$. Since h is continuous on $[0, x]$ and differentiable on $(0, x)$, by the Mean Value Theorem there exists $z \in (0, x)$ such that

$$h(x) - h(0) = h'(z)(x - 0)$$

Since $x > 0$, $h(0) = 0$, and $h'(z) \geq 0$, this implies

$$h(x) = h'(z)x \geq 0$$

Since $x > 0$ was arbitrary, this implies $h(x) \geq 0$ for all $x \geq 0$. Thus $h(x) = g(x) - f(x) \geq 0$ for all $x \geq 0$, or $f(x) \leq g(x)$ for all $x \geq 0$.

5. (30) Let $X \subseteq \mathbb{R}^n$ and $f, g : X \to \mathbb{R}^m$ be continuous functions. Let $\Psi : X \to 2^{\mathbb{R}^m}$ be a correspondence such that for each $x \in X$,

$$\Psi(x) = \{tf(x) + (1-t)g(x) : t \in [0,1]\}$$

Show that Ψ is upper hemi-continuous.

Solution: First let $h : \mathbb{R} \times X \to \mathbb{R}^m$ be given by

$$h(t,x) = tf(x) + (1-t)g(x)$$

Then note that since f and g are continuous, h is continuous on $\mathbb{R} \times X$ (using the standard Euclidean metric on $\mathbb{R}^{n+1}$). Next note that for each $x \in X$,

$$\Psi(x) = \{tf(x) + (1-t)g(x) : t \in [0,1]\} = h([0,1] \times \{x\})$$

Then let $x_0 \in X$. Let $V \subseteq \mathbb{R}^m$ be an open set such that $\Psi(x_0) \subseteq V$. Since h is continuous and V is open, $h^{-1}(V) \subseteq \mathbb{R} \times X$ is open. Then

$$\Psi(x_0) = h([0,1] \times \{x_0\}) \subseteq V \Rightarrow [0,1] \times \{x_0\} \subseteq h^{-1}(V)$$

Since $h^{-1}(V)$ is open and $[0,1] \times \{x_0\}$ is compact, there exists $\varepsilon > 0$ such that

$$B_\varepsilon([0,1] \times \{x_0\}) \subseteq h^{-1}(V)$$

(for example, by #5 on the 2019 exam). Then let $U = B_\varepsilon(x_0)$. Then $x_0 \in U$ and U is open. For $x \in U$ and $t \in [0,1]$, $(t,x) \in B_\varepsilon([0,1] \times \{x_0\}) \subseteq h^{-1}(V)$, so

$$h(t,x) = tf(x) + (1-t)g(x) \in V$$

Thus

$$\Psi(x) = \{tf(x) + (1-t)g(x) : t \in [0,1]\} \subseteq V \quad \forall x \in U$$

Thus Ψ is uhc at x_0. Since $x_0 \in X$ was arbitrary, Ψ is uhc.

Here is an argument instead using the sequential characterization of uhc. First note that for each $x \in X$, $\Psi(x)$ is compact. This is straightforward to show. For example, $[0,1] \times \{x\} \subseteq \mathbb{R} \times \mathbb{R}^n$ is a compact set, and since h is continuous, $h([0,1] \times \{x\}) = \Psi(x)$ is a compact set. Alternatively, it is straightforward to show this directly. For example, since $\Psi(x) \subseteq \mathbb{R}^m$ it suffices to show that $\Psi(x)$ is closed and bounded. To see that $\Psi(x)$ is bounded, note that if $y \in \Psi(x)$ then $y = tf(x) + (1-t)g(x)$ for some $t \in [0,1]$, so

$$\|y\| = \|tf(x) + (1-t)g(x)\| \leq t\|f(x)\| + (1-t)\|g(x)\| \leq \|f(x)\| + \|g(x)\|$$

To see that $\Psi(x)$ is closed, suppose $\{y_n\} \subseteq \Psi(x)$ and $y_n \to y$ for some $y \in \mathbb{R}^m$. For each n, $y_n = t_n f(x) + (1-t_n)g(x)$ for some $t_n \in [0,1]$. Then $\{t_n\} \subseteq [0,1]$ and $[0,1]$ is compact, so there is a subsequence $\{t_{n_k}\}$ such that $t_{n_k} \to t \in [0,1]$. Then $y_{n_k} = t_{n_k} f(x) + (1-t_{n_k})g(x) \to tf(x) + (1-t)g(x)$, and also $y_{n_k} \to y$, so $y = tf(x) + (1-t)g(x)$. This implies $y \in \Psi(x)$ by definition, which shows that $\Psi(x)$ is closed.

Since $\Psi(x)$ is compact for each $x \in X$, the sequential characterization of uhc can be used to show that Ψ is uhc. Then to that end, let $x_0 \in X$. Suppose $\{x_n\} \subseteq X$ with $x_n \to x_0$, and $y_n \in \Psi(x_n)$ for each n. By definition, for every n there exists $t_n \in [0,1]$ such that

$$y_n = t_n f(x_n) + (1-t_n)g(x_n)$$

Then $\{t_n\} \subseteq [0,1]$ and $[0,1]$ is compact, so there is a subsequence $\{t_{n_k}\}$ such that $t_{n_k} \to t \in [0,1]$. Thus using the continuity of f and g,

$$y_{n_k} = t_{n_k} f(x_{n_k}) + (1-t_{n_k})g(x_{n_k}) \to tf(x_0) + (1-t)g(x_0)$$

and $y = tf(x_0) + (1-t)g(x_0) \in \Psi(x_0)$ by definition. Thus Ψ is uhc at x_0. Since $x_0 \in X$ was arbitrary, Ψ is uhc.

6. (30) Let (X,d) be a metric space and $F_i \subseteq X$ be compact for each $i \in \mathbb{N}$. Let $U \subseteq X$ be an open set such that $\bigcap_{i=1}^{\infty} F_i \subseteq U$. Show that there exists $n \in \mathbb{N}$ such that $\bigcap_{i=1}^{n} F_i \subseteq U$.

Solution: Suppose not. Then for every $n \in \mathbb{N}$, $\bigcap_{i=1}^{n} F_i \not\subseteq U$. So for every n there exists $x_n \in (\bigcap_{i=1}^{n} F_i) \setminus U$. Since $x_n \in \bigcap_{i=1}^{n} F_i$ for each n, $\{x_n\} \subseteq F_1$. Then F_1 is compact, so there is a subsequence $\{x_{n_k}\}$ such that $x_{n_k} \to x \in F_1$. For each n there exists $K > 0$ such that $n_k > n$ for all $k > K$. Thus given n, there exists $K > 0$ such that

$$x_{n_k} \in \bigcap_{i=1}^{n_k} F_i \subseteq \bigcap_{i=1}^{n} F_i \quad \forall k > K$$

Since $\bigcap_{i=1}^{n} F_i$ is closed, this implies $x \in \bigcap_{i=1}^{n} F_i$. This is true for every n, so $x \in \bigcap_{i=1}^{\infty} F_i$. Then $\bigcap_{i=1}^{\infty} F_i \subseteq U$ by assumption, so $x \in U$ where U is open. But $x_{n_k} \notin U$ for each k by construction, which is a contradiction. Thus there exists $n \in \mathbb{N}$ such that $\bigcap_{i=1}^{n} F_i \subseteq U$.

Here is another argument using open covers. By assumption $\bigcap_{i=1}^{\infty} F_i \subseteq U$, which implies $(\bigcap_{i=1}^{\infty} F_i) \setminus U = \bigcap_{i=1}^{\infty}(F_i \setminus U) = \emptyset$. Then note that for each i, $F_i \setminus U = F_i \cap U^c$, and U^c is closed since U is open. Thus $F_i \setminus U$ is compact for each i. Now consider $G = F_1 \setminus U$. Then G is compact, and

$$\emptyset = \bigcap_{i=1}^{\infty}(F_i \setminus U) = (F_1 \setminus U) \cap \left(\bigcap_{i=2}^{\infty}(F_i \setminus U)\right) = G \cap \left(\bigcap_{i=2}^{\infty}(F_i \setminus U)\right)$$

This implies

$$G = F_1 \setminus U \subseteq \left(\bigcap_{i=2}^{\infty}(F_i \setminus U)\right)^c = \bigcup_{i=2}^{\infty}(F_i \setminus U)^c$$

Let $V_i = (F_i \setminus U)^c$ for each $i = 2, 3, \ldots$. Then V_i is open for each i, and $G \subseteq \bigcup\limits_{i=2}^{\infty} V_i$, so $\{V_i : i = 2, 3, \ldots\}$ is an open cover of G. Since G is compact, there exist $i_1, \ldots, i_m$ such that $G \subseteq V_{i_1} \cup \cdots \cup V_{i_m}$. Set $n = \max\{i_1, \ldots, i_m\}$, and note that

$$V_{i_1} \cup \cdots \cup V_{i_m} \subseteq \bigcup_{i=2}^{n} V_i$$

So

$$G \subseteq V_{i_1} \cup \cdots \cup V_{i_m} \subseteq \bigcup_{i=2}^{n} V_i = \bigcup_{i=2}^{n}(F_i \setminus U)^c$$

Then $G = F_1 \setminus U \subseteq \bigcup\limits_{i=2}^{n}(F_i \setminus U)^c = (\bigcap\limits_{i=2}^{n}(F_i \setminus U))^c$, which implies $F_1 \setminus U \cap (\bigcap\limits_{i=2}^{n}(F_i \setminus U)) = \emptyset$. Thus $\bigcap\limits_{i=1}^{n}(F_i \setminus U) = (\bigcap\limits_{i=1}^{n} F_i) \setminus U = \emptyset$. This implies $\bigcap\limits_{i=1}^{n} F_i \subseteq U$.

7. (30) Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a continuous function. Let

$$B = \{x \in \mathbb{R}^n : x = \lambda f(x) \text{ for some } \lambda \in [0, 1]\}$$

Suppose B is bounded. Show that f has a fixed point.

(**Hint:** Choose $M > 0$ such that $\|x\| < M$ for all $x \in B$. If x^* is a fixed point of f, then $x^* = \lambda f(x^*)$ for $\lambda = 1$.)

Solution: First choose $M > 0$ such that $\|x\| < M$ for all $x \in B$; this is possible because B is bounded by assumption. Then define $g : \mathbb{R}^n \to \mathbb{R}^n$ by

$$g(x) = \begin{cases} f(x) & \text{if } \|f(x)\| \leq M \\ \frac{M}{\|f(x)\|} f(x) & \text{if } \|f(x)\| > M \end{cases}$$

Then note that $\|g(x)\| \leq M$ for all $x \in \mathbb{R}^n$, as if $\|f(x)\| \leq M$ then $g(x) = f(x)$, and thus $\|g(x)\| \leq M$ as well, and if $\|f(x)\| > M$ then $g(x) = \frac{M}{\|f(x)\|} f(x)$, so

$$\|g(x)\| = \left\| \frac{M}{\|f(x)\|} f(x) \right\| = \frac{M}{\|f(x)\|} \|f(x)\| = M$$

Thus $g(x) \in B_M(0)$ for all $x \in \mathbb{R}^n$. In particular, $g : B_M(0) \to B_M(0)$. Also note that if x^* is a fixed point of g, then x^* must also be a fixed point of f. To see this, suppose $g(x^*) = x^*$. If $\|f(x^*)\| \leq M$, then $g(x^*) = f(x^*)$ by definition, so $x^* = g(x^*) = f(x^*)$. If instead $\|f(x^*)\| > M$, then

$$x^* = g(x^*) = \frac{M}{\|f(x^*)\|} f(x^*)$$

Since $\|f(x^*)\| > M$, $\frac{M}{\|f(x^*)\|} < 1$. So $x^* = \lambda f(x^*)$ with $\lambda \in [0, 1]$, which implies $x^* \in B$ by definition. But

$$\|x^*\| = \|g(x^*)\| = \frac{M}{\|f(x^*)\|} \|f(x^*)\| = M$$

which is a contradiction, since $\|x\| < M$ for all $x \in B$ by construction.

Thus to show that f has a fixed point, it suffices to show that g has a fixed point. Then note that $B_M(0)$ is a nonempty, compact, convex subset of $\mathbb{R}^n$, and $g : B_M(0) \to B_M(0)$, so by Brouwer's Fixed Point Theorem, to show that g has a fixed point in $B_M(0)$ it suffices to show that g is continuous. To that end, let $x \in \mathbb{R}^n$ and suppose $\{x_n\} \subseteq \mathbb{R}^n$ with $x_n \to x$. If $\|f(x)\| < M$, then $g(x) = f(x)$. Using the continuity of f and of the norm $\|\cdot\|$, $\|f(x_n)\| < M$ for all n sufficiently large, so $g(x_n) = f(x_n)$ for all n sufficiently large. Thus

$$\lim_{n\to\infty} g(x_n) = \lim_{n\to\infty} f(x_n) = f(x) = g(x)$$

again using the continuity of f. Similarly, if $\|f(x)\| > M$, then $\|f(x_n)\| > M$ for all n sufficiently large, so without loss of generality suppose $\|f(x_n)\| > 0$ for all n. Then in this case, $g(x) = \frac{M}{\|f(x)\|} f(x)$ and $g(x_n) = \frac{M}{\|f(x_n)\|} f(x_n)$ for all n sufficiently large. Thus

$$\lim_{n\to\infty} g(x_n) = \lim_{n\to\infty} \frac{M}{\|f(x_n)\|} f(x_n) = \frac{M}{\|f(x)\|} f(x) = g(x)$$

Finally, if $\|f(x)\| = M$, then $g(x) = f(x)$. For all n sufficiently large, $\|f(x_n)\| > \frac{M}{2} > 0$, so without loss of generality suppose $\|f(x_n)\| > 0$ for all n. Then

$$g(x_n) = \min\left(1, \frac{M}{\|f(x_n)\|}\right) f(x_n) \to f(x) = g(x)$$

Thus g is continuous at x. Since $x \in \mathbb{R}^n$ was arbitrary, g is continuous.

By Brouwer's Fixed Point Theorem, g has a fixed point in $B_M(0)$. Thus there exists $x^* \in B_M(0)$ such that $x^* = g(x^*)$. By the argument above, x^* is also a fixed point of f.