Economics 204 Summer/Fall 2020
Final Exam

Answer all of the questions below. Be as complete, correct, and concise as possible. There are 7 questions for a total of 180 points possible; point values for each problem are in parentheses. For questions with subparts, each subpart is worth the same number of points. Use the points as a guide to allocating your time.

1. (15) Let A and B be $n \times n$ matrices that are similar, so there exists an invertible $n \times n$ matrix P such that $A = P^{-1}BP$. Show that for every $k \in \mathbb{N}$, $A^k = P^{-1}B^kP$ (where M^k is the product of k copies of the $n \times n$ matrix M).

   (**Hint:** use induction.)

2. (15) Let (X, d) and (Y, ρ) be metric spaces and $f, g : X \to Y$ be continuous functions. Let $E \subseteq X$ be a dense subset of X, that is, a set such that $\overline{E} = X$. Show that if $f(z) = g(z)$ for all $z \in E$, then $f = g$, that is, $f(x) = g(x)$ for all $x \in X$.

3. (30) Let X and Y be vector spaces over the same field F, and let $T : X \to Y$ be a linear transformation. Suppose $W \subseteq X$ is a subset of X that spans X, and $T(W) \subseteq Y$ is linearly independent.

   (a.) Show that T is one-to-one.

   (b.) Show that W is a basis for X.

4. (30) Let $f, g : \mathbb{R} \to \mathbb{R}$ be differentiable functions. Suppose that $f(0) = g(0)$ and $f'(x) \leq g'(x)$ for all $x \in \mathbb{R}$. Show that $f(x) \leq g(x)$ for all $x \geq 0$.

5. (30) Let $X \subseteq \mathbb{R}^n$ and $f, g : X \to \mathbb{R}^m$ be continuous functions. Let $\Psi : X \to 2^{\mathbb{R}^m}$ be a correspondence such that for each $x \in X$,

$$\Psi(x) = \{tf(x) + (1-t)g(x) : t \in [0,1]\}$$

Show that Ψ is upper hemi-continuous.

6. (30) Let (X, d) be a metric space and $F_i \subseteq X$ be compact for each $i \in \mathbb{N}$. Let $U \subseteq X$ be an open set such that $\bigcap_{i=1}^{\infty} F_i \subseteq U$. Show that there exists $n \in \mathbb{N}$ such that $\bigcap_{i=1}^{n} F_i \subseteq U$.

7. (30) Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a continuous function. Let

$$B = \{x \in \mathbb{R}^n : x = \lambda f(x) \text{ for some } \lambda \in [0,1]\}$$

Suppose B is bounded. Show that f has a fixed point.

(**Hint:** Choose $M > 0$ such that $\|x\| < M$ for all $x \in B$. If x^* is a fixed point of f, then $x^* = \lambda f(x^*)$ for $\lambda = 1$.)