Econ 204 2021

Lecture 15

Outline

1. Second Order Linear Differential Equations
2. Inhomogeneous Linear Differential Equations
3. Nonlinear Differential Equations - Linearization

Announcements

- PS 5 due today 1pm
  ⇝ solns posted ≈ 2 pm
- PS 6 due Monday 1 pm
  ⇝ solns posted ≈ 2pm
- posted corrected lecture 14 slides

Higher Order Differential Equations

A differential equation *of order* n is an equation of the form

$$y^{(n)}(t) = F(y(t), y'(t), \ldots, y^{(n-1)}(t), t) \qquad (*)$$

$F: \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$, $y(t) \in \mathbb{R} \ \forall t$

We can always rewrite an n^{th} order equation as a system of n first-order equations by redefining variables.

Set $\bar{y} = \begin{pmatrix} y \\ y' \\ \vdots \\ y^{(n-1)} \end{pmatrix}$

Rewrite $(*)$:

$$\bar{y}'(t) = \bar{F}(\bar{y}(t), t)$$

for appropriately chosen $\bar{F}: \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$

Second Order Linear Differential Equations

Consider the second order differential equation $y'' = cy + by'$ with $b, c \in \mathbf{R}$.

Rewrite this as a **first order** linear differential equation in two variables:

Define

$$\bar{y}(t) = \begin{pmatrix} y(t) \\ y'(t) \end{pmatrix}$$

$y''(t) = cy(t) + by'(t)$

Then

$$\begin{aligned} \bar{y}'(t) &= \begin{pmatrix} y'(t) \\ y''(t) \end{pmatrix} \\ &= \begin{pmatrix} 0 & 1 \\ c & b \end{pmatrix} \begin{pmatrix} y(t) \\ y'(t) \end{pmatrix} = \begin{pmatrix} y'(t) \\ cy(t) + by'(t) \end{pmatrix} \\ &= \begin{pmatrix} 0 & 1 \\ c & b \end{pmatrix} \bar{y}(t) \end{aligned}$$

The eigenvalues are the roots of the equation $\lambda^2 - b\lambda - c = 0$, which are $\frac{b \pm \sqrt{b^2+4c}}{2}$.

The qualitative behavior of the solutions can be explicitly described from the coefficients b and c, by determining whether the eigenvalues are real or complex, and whether the real parts are negative, zero, or positive. (if coefficient matrix is diagonalizable)

yesterday

Example Consider the second order linear differential equation

$$y'' = 2y + y'$$

As above, let

$$\bar{y} = \begin{pmatrix} y \\ y' \end{pmatrix}$$

so the equation becomes

$$\bar{y}' = \begin{pmatrix} 0 & 1 \\ 2 & 1 \end{pmatrix} \bar{y}$$

The eigenvalues are the roots of the characteristic polynomial

$$\lambda^2 - \lambda - 2 = 0$$

Eigenvalues and corresponding eigenvectors are given by

$$\begin{array}{ll} \lambda_1 = 2 & v_1 = (1, 2) \\ \lambda_2 = -1 & v_2 = (1, -1) \end{array}$$

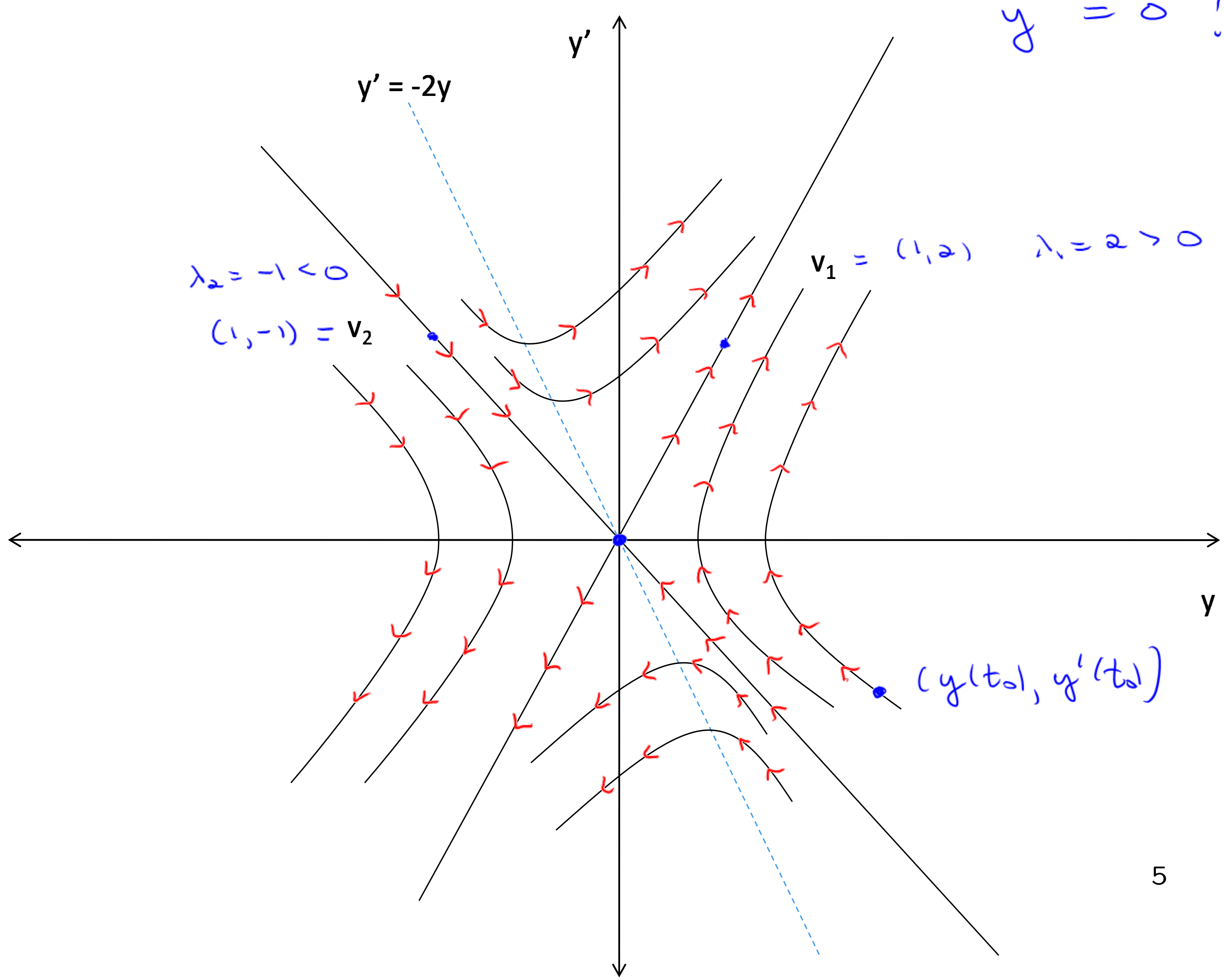
$n = 2$ distinct eigenvalues $\Rightarrow$ diagonalizable!

From this information alone, we know the qualitative properties of the solutions are as given in the phase plane diagram

$y'' = 2y + y'$

$y'' = 2y + y'$

$y'' = 0$?



- Solutions are roughly hyperbolic in shape with asymptotes along the eigenvectors. Along the eigenvector v_1, the solutions flow off to infinity; along the eigenvector v_2, the solutions converge to zero.

- Solutions flow in directions consistent with flows along asymptotes

- On the y-axis, we have $y' = 0$, which means that everywhere on the y-axis (except at the stationary point 0), the solution must have a vertical tangent.

- On the y'-axis, we have $y = 0$, so we have

$$y'' = 2y + y' = y'$$

Thus, above the y-axis, $y'' = y' > 0$, so y' is increasing along the direction of the solution; below the y-axis, $y'' = y' < 0$, so y' is decreasing along the direction of the solution.

- Along the line $y' = -2y$, $y'' = 2y - 2y = 0$, so y' achieves a minimum or maximum where it crosses that line.

The general solution is given by

$$\begin{pmatrix} y(t) \\ y'(t) \end{pmatrix} = Mtx_{U,V}(id) \begin{pmatrix} e^{2(t-t_0)} & 0 \\ 0 & e^{-(t-t_0)} \end{pmatrix} Mtx_{V,U}(id) \begin{pmatrix} y(t_0) \\ y'(t_0) \end{pmatrix}$$

eigenvectors inverse

$$\begin{aligned}
&= \begin{pmatrix} 1 & 1 \\ 2 & -1 \end{pmatrix} \begin{pmatrix} e^{2(t-t_0)} & 0 \\ 0 & e^{-(t-t_0)} \end{pmatrix} \begin{pmatrix} 1/3 & 1/3 \\ 2/3 & -1/3 \end{pmatrix} \begin{pmatrix} y(t_0) \\ y'(t_0) \end{pmatrix} \\
&= \begin{pmatrix} \frac{e^{2(t-t_0)}+2e^{-(t-t_0)}}{3} & \frac{e^{2(t-t_0)}-e^{-(t-t_0)}}{3} \\ \frac{2e^{2(t-t_0)}-2e^{-(t-t_0)}}{3} & \frac{2e^{2(t-t_0)}+e^{-(t-t_0)}}{3} \end{pmatrix} \begin{pmatrix} y(t_0) \\ y'(t_0) \end{pmatrix} \\
&= \begin{pmatrix} \frac{y(t_0)+y'(t_0)}{3} e^{2(t-t_0)} + \frac{2y(t_0)-y'(t_0)}{3} e^{-(t-t_0)} \\ \frac{2y(t_0)+2y'(t_0)}{3} e^{2(t-t_0)} + \frac{-2y(t_0)+y'(t_0)}{3} e^{-(t-t_0)} \end{pmatrix}
\end{aligned}$$

The general solution has two real degrees of freedom; a specific solution is determined by specifying initial conditions $y(t_0)$ and $y'(t_0)$.

Because

$$\bar{y} = \begin{pmatrix} y \\ y' \end{pmatrix}$$

it is easier to find the general solution by setting

$$y(t) = C_1 e^{2(t-t_0)} + C_2 e^{-(t-t_0)}$$

$$\Rightarrow \quad y'(t) = 2C_1 e^{2(t-t_0)} - C_2 e^{-(t-t_0)}$$

Then

$$\begin{aligned}
\boxed{y(t_0) = C_1 + C_2} \\
y'(t) &= 2C_1 e^{2(t-t_0)} - C_2 e^{-(t-t_0)} \\
\boxed{y'(t_0) = 2C_1 - C_2} \\
C_1 &= \frac{y(t_0) + y'(t_0)}{3} \\
C_2 &= \frac{2y(t_0) - y'(t_0)}{3} \\
y(t) &= \frac{y(t_0) + y'(t_0)}{3} e^{2(t-t_0)} + \frac{2y(t_0) - y'(t_0)}{3} e^{-(t-t_0)}
\end{aligned}$$

$y'(t) = M(t)\, y(t)$

$M(t)(cy)(t) = c\,M(t)y(t)$
$= c\,y'(t)$
$= (cy)'(t)$

Inhomogeneous Linear Differential Equations with Nonconstant Coefficients

Consider the inhomogeneous linear differential equation

$$y' = M(t)y + H(t) \tag{1}$$

where M is continuous function from t to the set of $n \times n$ matrices; and H is continuous function from t to $\mathbf{R}^n$.

$M : \mathbb{R} \to L(\mathbb{R}^n, \mathbb{R}^n)$
$H : \mathbb{R} \to \mathbb{R}^n$

There is a close relationship between solutions of the *inhomogeneous* linear differential equation (1) and the associated *homogeneous* linear differential equation

$$y' = M(t)y \tag{2}$$

homogeneous: if y is a solution given y_0, then cy is a solution given cy_0 for any $c \in \mathbb{R}$

Inhomogeneous Linear Differential Equations with Nonconstant Coefficients

Theorem 1. *The general solution of the inhomogeneous linear differential equation (1) is*

$$y_h + y_p$$

where y_h is the general solution of the homogeneous linear differential equation (2) and y_p is any particular solution of the inhomogeneous linear differential equation (1).

Proof. Fix any particular solution y_p of inhomogeneous equation (1). Suppose y_h is any solution of the corresponding homogeneous equation (2). Let $y_i(t) = y_h(t) + y_p(t)$.

$$\begin{aligned}
y_i'(t) &= y_h'(t) + y_p'(t) \\
&= M(t)y_h(t) + M(t)y_p(t) + H(t) \\
&= M(t)(y_h(t) + y_p(t)) + H(t) \\
&= M(t)y_i(t) + H(t)
\end{aligned}$$

so y_i is solution of inhomogeneous equation (1).

Conversely, suppose y_i is any solution of inhomogenous equation (1). Let $y_h(t) = y_i(t) - y_p(t)$.

$$\begin{aligned}
y_h'(t) &= y_i'(t) - y_p'(t) \\
&= M(t)y_i(t) + H(t) - M(t)y_p(t) - H(t) \\
&= M(t)(y_i(t) - y_p(t)) \\
&= M(t)y_h(t)
\end{aligned}$$

so y_h is solution of homogeneous equation (2) and $y_i = y_h + y_p$.

$\square$

Remark: To find general solution of inhomogeneous equation:

1. Find general solution of homogeneous equation;
2. Find a particular solution of inhomogeneous equation;
3. Add these to get general solution of inhomogeneous equation

One nice case in which we can easily give closed form expression:

Theorem 2. *Consider the inhomogeneous linear differential equation (1), and suppose that $M(t)$ is a constant matrix M, independent of t. A particular solution of the inhomogeneous linear differential equation (1), satisfying the initial condition $y_p(t_0) = y_0$, is given by*

$$y_p(t) = e^{(t-t_0)M} y_0 + \int_{t_0}^{t} e^{(t-s)M} H(s)\, ds \tag{3}$$

The general solution to the corresponding homogeneous equation $y'(t) = My(t)$ is

$$y_h(t) = e^{(t-t_0)M} y_0 \,, \quad y_0 \in \mathbb{R}^n$$

Recall: $x \in \mathbb{R}$ or $\mathbb{C}$: $e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!} = \lim_{n\to\infty} \sum_{k=0}^{n} \frac{x^k}{k!}$

Matrix Exponentials

$\sum_{k=0}^{n} z_k$

Here for an $n \times n$ matrix M, we define

(define $M^0 = I$)

$$e^M = \sum_{k=0}^{\infty} \frac{M^k}{k!} = I + M + \frac{M^2}{2} + \cdots = \lim_{n\to\infty} \sum_{k=0}^{n} \frac{M^k}{k!}$$

$n \times n$ matrix

and

$$e^{tM} = \sum_{k=0}^{\infty} \frac{t^k M^k}{k!} \qquad t \in \mathbb{R}$$

- if D is a diagonal matrix with diagonal $d_1, \ldots, d_n$,

$$e^D = \begin{pmatrix} e^{d_1} & 0 & \cdots & 0 \\ 0 & e^{d_2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & e^{d_n} \end{pmatrix}$$

Recall: $x, y \in \mathbb{C}$ or $\mathbb{R}$
$e^{x+y} = e^x e^y$

- $e^{A+B} = e^A e^B$ if $AB = BA$ (not necessarily if A and B do not commute)

- $e^{P^{-1}AP} = P^{-1}e^A P$
  ($\Rightarrow$ solution for M diagonalizable immediate from general homogeneous solution)

- $g(t) = e^{tM}$ is differentiable and $g'(t) = Me^{tM}$
  (C^∞)

$e^{(t-s)M} = e^{(t-t_0)M} e^{-(s-t_0)M}$

Proof. We verify that y_p solves (1):

$$
\begin{aligned}
y_p(t) &= e^{(t-t_0)M} y_0 + \int_{t_0}^{t} e^{(t-s)M} H(s)\, ds \\
&= e^{(t-t_0)M} y_0 + \int_{t_0}^{t} e^{(t-t_0)M} e^{-(s-t_0)M} H(s)\, ds \\
&= e^{(t-t_0)M} \left(y_0 + \int_{t_0}^{t} e^{-(s-t_0)M} H(s)\, ds \right) \longleftarrow \text{pull out } e^{(t-t_0)M} \\
y_p'(t) &= M e^{(t-t_0)M} \left(y_0 + \int_{t_0}^{t} e^{-(s-t_0)M} H(s)\, ds \right) \\
&\quad + e^{(t-t_0)M} \left(e^{-(t-t_0)M} H(t) \right) \longleftarrow \text{chain rule + Fund. Thm. of Calculus} \\
&= M y_p(t) + H(t) \\
y_p(t_0) &= e^{(t_0-t_0)M} y_0 + \int_{t_0}^{t_0} e^{(s-t_0)M} H(s)\, ds \\
&= y_0
\end{aligned}
$$

$\square$

Example Consider the inhomogeneous linear differential equation

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}' = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} + \begin{pmatrix} \sin t \\ \cos t \end{pmatrix}$$

$t_0 = 0$

$y_0 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

By Theorem 2, a particular solution is given by

$H(s)$

$$\begin{aligned}
y_p(t) &= e^{(t-t_0)M} y_0 + \int_{t_0}^{t} e^{(t-s)M} H(s)\, ds \\
&= \underbrace{\begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix}}_{e^{tM}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \int_0^t \begin{pmatrix} e^{(t-s)} & 0 \\ 0 & e^{-(t-s)} \end{pmatrix} \begin{pmatrix} \sin s \\ \cos s \end{pmatrix} ds \\
&= \begin{pmatrix} e^t \\ e^{-t} \end{pmatrix} + \int_0^t \begin{pmatrix} e^{t-s} \sin s \\ e^{s-t} \cos s \end{pmatrix} ds \\
&= \begin{pmatrix} e^t \left(1 + \int_0^t e^{-s} \sin s\, ds\right) \\ e^{-t} \left(1 + \int_0^t e^{s} \cos s\, ds\right) \end{pmatrix}
\end{aligned}$$

Thus, the general solution of the original inhomogeneous equation is given by

general solution to homogeneous

particular solution to inhomogeneous

$$\begin{aligned}
\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} &= \begin{pmatrix} C_1 e^t \\ C_2 e^{-t} \end{pmatrix} + \begin{pmatrix} e^t \left(1 + \int_0^t e^{-s} \sin s \, ds\right) \\ e^{-t} \left(1 + \int_0^t e^{s} \cos s \, ds\right) \end{pmatrix} \\
&= \begin{pmatrix} D_1 e^t - \frac{\sin t + \cos t}{2} \\ D_2 e^{-t} + \frac{\sin t + \cos t}{2} \end{pmatrix} \quad \text{(after much simplification)}
\end{aligned}$$

where D_1 and D_2 are arbitrary real constants.

Phase diagram for solutions to homogeneous equation

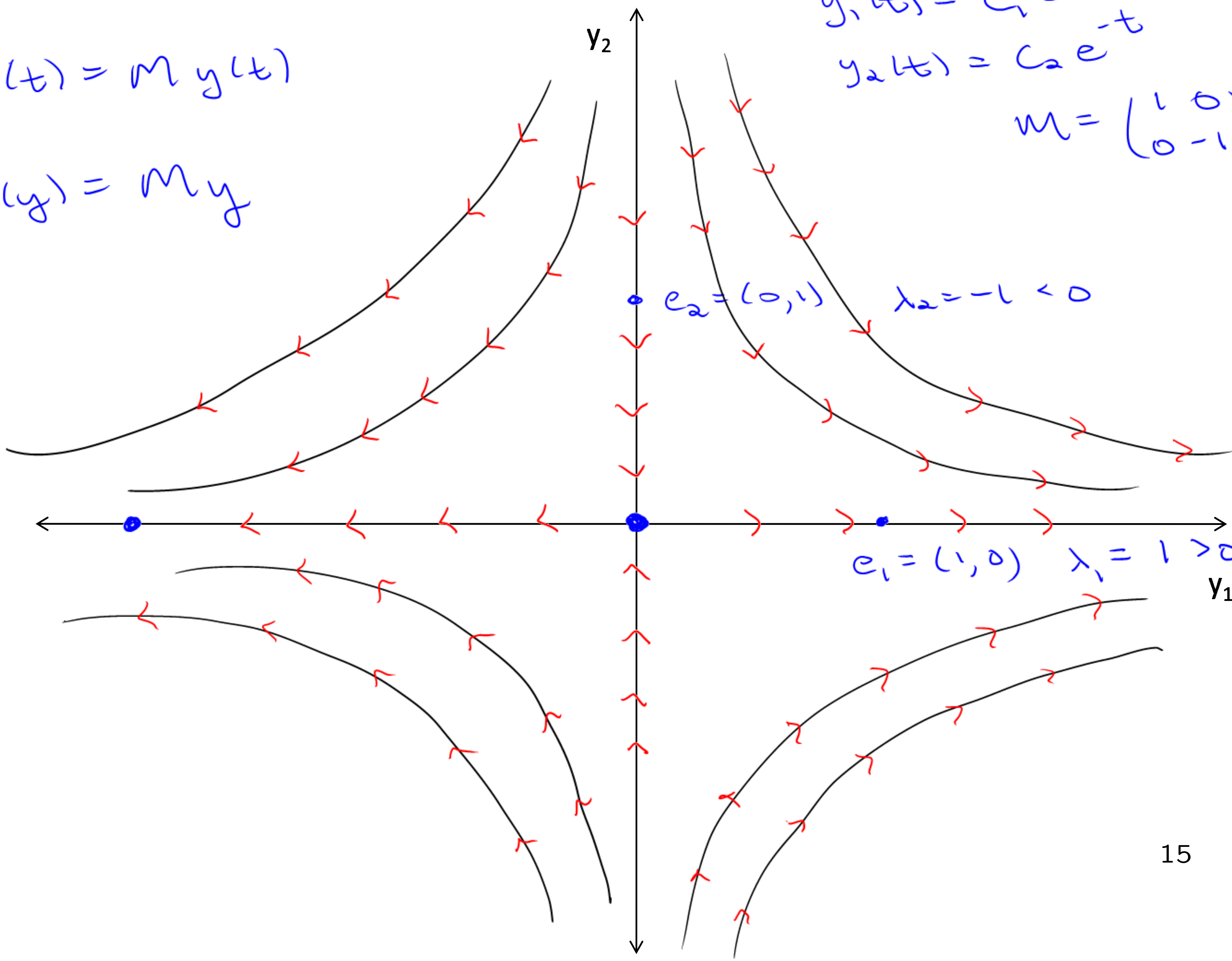
$y'(t) = M\,y(t)$

$F(y) = My$

$y_1(t) = c_1 e^{t}$

$y_2(t) = c_2 e^{-t}$

$M = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$



Nonlinear Differential Equations - Linearization

- Nonlinear differential equations are very difficult to solve in closed form.

- Specific techniques solve special classes of equations.

- Numerical methods compute numerical solutions of any ordinary differential equation.

- Linearization can provide qualitative information about the solutions of nonlinear autonomous equations.

Nonlinear Differential Equations - Stability

Linearization provides information about qualitative properties of solutions of nonlinear differential equations near the stationary points. Suppose y_s is a stationary point:

$y(t)$ solution to nonlinear problem $y'(t) = F(y(t))$

- If eigenvalues of linearized equation at y_s all have strictly negative real parts, there exists $\varepsilon > 0$ such that $|y(0) - y_s| < \varepsilon \Rightarrow \lim_{t\to\infty} y(t) = y_s$. All solutions of the original nonlinear equation which start sufficiently close to the stationary point y_s converge to y_s.

- If eigenvalues of the linearized equation at y_s all have strictly positive real parts, no solution of the original nonlinear equation converges to y_s.

- If eigenvalues of the linearized equation at y_s all have real part zero, then the solutions of linearized equation are closed curves around y_s. This tells us little about the solutions of nonlinear equation. They may

  - follow closed curves around y_s

  - converge to y_s

  - converge to a limit closed curve around y_s

  - diverge from y_s

  - converge to y_s along certain directions and diverge from y_s along other directions.

Example: Pendulum The equation of motion of a frictionless pendulum is a nonlinear autonomous differential equation

$$y'' = -\alpha^2 \sin y,\ \alpha > 0$$

Here, y is the angle between the pendulum and a vertical line. The fact that the motion follows this differential equation is obtained by resolving the downward force of gravity into two components, one tangent to the curve the pendulum follows and one which is parallel to the pendulum; the latter component is canceled by the pendulum rod.

$y'' = -\alpha^2 \sin y$

Transform to first-order equation:

Define

$$\bar{y}(t) = \begin{pmatrix} y(t) \\ y'(t) \end{pmatrix} = \begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix}$$

so differential equation becomes

$$\bar{y}'(t) = \begin{pmatrix} y_2(t) \\ -\alpha^2 \sin y_1(t) \end{pmatrix} = \begin{pmatrix} y'(t) \\ y''(t) \end{pmatrix}$$

Let

$$F(\bar{y}) = \begin{pmatrix} y_2 \\ -\alpha^2 \sin y_1 \end{pmatrix}$$

$F : \mathbb{R}^2 \to \mathbb{R}^2$

$\bar{y}'(t) = F(\bar{y}(t))$

Solve for stationary points: points $\bar{y}$ such that $F(\bar{y}) = 0$:

$$
\begin{aligned}
F(\bar{y}) = 0 &\iff \begin{pmatrix} y_2 \\ -\alpha^2 \sin y_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\
&\iff \sin y_1 = 0 \text{ and } y_2 = 0 \\
&\iff y_1 = n\pi \text{ and } y_2 = 0
\end{aligned}
$$

so set of stationary points is

$$\{(n\pi, 0) : n \in \mathbf{Z}\}$$

Linearize the equation around each of the stationary points:

$$F(y_1, y_2) = \begin{pmatrix} y_2 \\ -\alpha^2 \sin y_1 \end{pmatrix}$$

Take the first order Taylor polynomial for F: at $(n\pi, 0)$:

evaluated at $(n\pi, 0)$

$$\begin{aligned}
F(n\pi + h, 0 + k) + o(|h| + |k|) &= \cancel{F(n\pi, 0)} + \begin{pmatrix} \frac{\partial F_1}{\partial y_1} & \frac{\partial F_1}{\partial y_2} \\ \frac{\partial F_2}{\partial y_1} & \frac{\partial F_2}{\partial y_2} \end{pmatrix} \begin{pmatrix} h \\ k \end{pmatrix} \\
&= \begin{pmatrix} 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ -\alpha^2 \cos n\pi & 0 \end{pmatrix} \begin{pmatrix} h \\ k \end{pmatrix} \\
&= \begin{pmatrix} 0 & 1 \\ \underbrace{(-1)^{n+1}\alpha^2}_{\cos n\pi = (-1)^n} & 0 \end{pmatrix} \begin{pmatrix} h \\ k \end{pmatrix}
\end{aligned}$$

- For n even, the eigenvalues are solutions to

$$\begin{aligned}
\lambda^2 + \alpha^2 &= 0 \\
\text{so } \lambda_1 &= i\alpha, \ \lambda_2 = -i\alpha
\end{aligned}$$

- $n = 2$ distinct eigenvalues $\Rightarrow$ diagonalizable!
- all real parts of eigenvalues $= 0$
  $\Rightarrow$ solutions spiral around stationary points

Close to $(n\pi, 0)$ for n even, the solutions spiral around the stationary point. For $y_2 = y_1' > 0$, y_1 is increasing, so the solutions move in a clockwise direction.

- For n odd, the eigenvalues solve $\lambda^2 - \alpha^2 = 0$, so the eigenvalues and eigenvectors are

$$\begin{aligned} \lambda_1 &= \alpha,\ \lambda_2 = -\alpha \\ v_1 &= (1, \alpha),\ v_2 = (1, -\alpha) \end{aligned}$$

$n=2$ distinct eigenvalues $\Rightarrow$ diagonalizable!

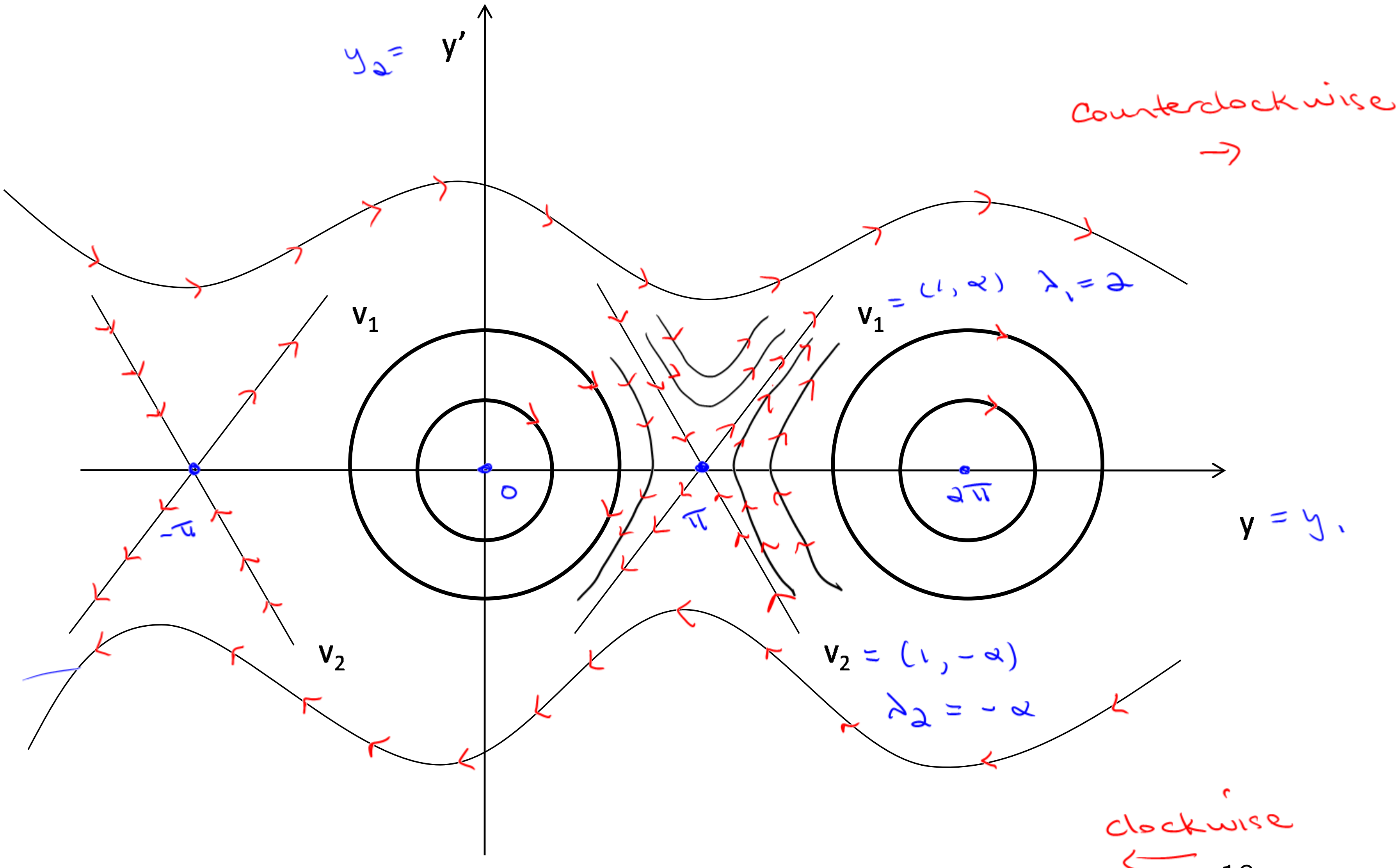
Close to $(n\pi, 0)$ for n odd, the solutions are roughly hyperbolic in shape; along v_2, they converge to the stationary point, while along v_1, they diverge from the stationary point. The solutions of the linearized equation tend to infinity along v_1. The stationary point $(n\pi, 0)$ with n odd corresponds to the pendulum pointing vertically upwards.

- From this information alone, we know the qualitative properties of the solutions of the linearized equation are as given in the phase plane diagram in Figure 2; the solutions of the original equation will closely follow these near the ~~stable~~ stationary points:

  - On the y-axis, we have $y' = 0$, which means that everywhere on the y-axis (except at the stationary points), the solution must have a vertical tangent.

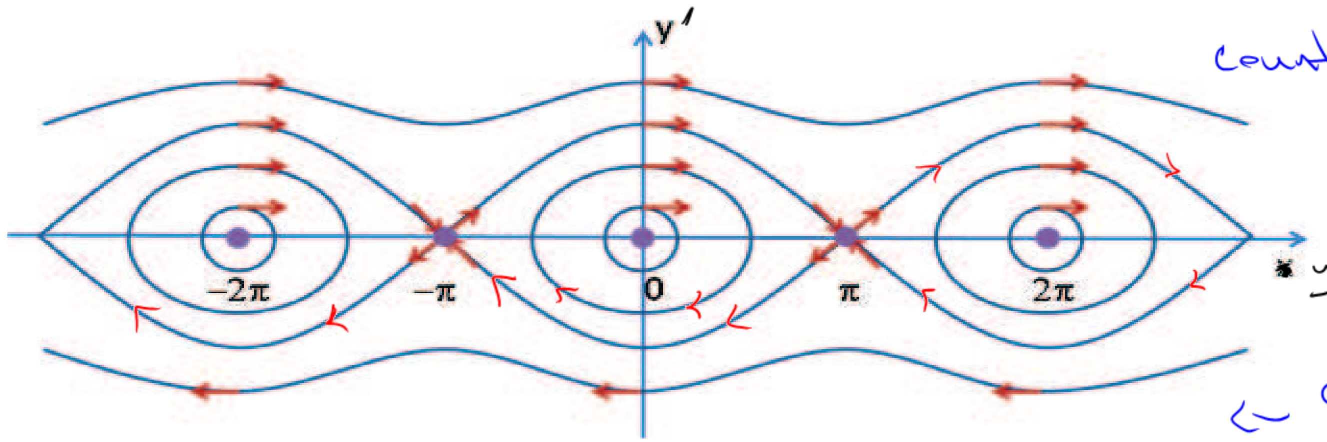
  - Solve $y'' = -\alpha^2 \sin y = 0$, so $y = n\pi$; thus, at $y = n\pi$, the derivative of y' is zero, so the tangent to the curve is horizontal.

- If the initial value of $|y_2|$ is sufficiently large, the solutions of the original equation no follow longer closed curves; this corresponds to the pendulum going "over the top" rather than oscillating back and forth.



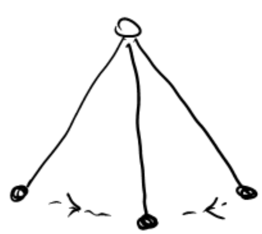
y₂ = y'
counterclockwise
v₁
v₁ = (1, α) λ₁ = α
0
π
-π
2π
y = y₁
v₂
v₂ = (1, −α)
λ₂ = −α
clockwise

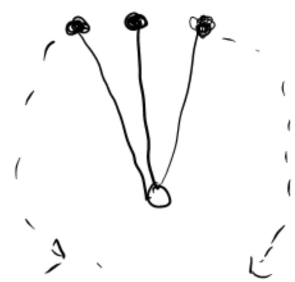
counterclockwise →



← clockwise

n even:





: n odd

$\{T_n\} \subseteq E^*$ Cauchy

$\forall n \in \mathbb{N}: T_n \in E^*$ $\quad T_n : E \to \mathbb{R}$

$\{\frac{1}{n}\}$ $\quad \frac{1}{n} \to 0$

$C = \bigcap_{n \in \mathbb{N}} \psi^n(X)$ $\quad$ compact, $\neq \emptyset$

$\psi(C) \subseteq C$

Show $C \subseteq \psi(C)$

$\forall y \in C, \quad y \in \psi^n(X) \quad \forall n$

Want to show $y \in \psi(C)$

$y \in C \Rightarrow y \in \psi^n(X) = \psi(\psi^{n-1}(X))$

$\Rightarrow \forall n \ \exists z_n \in \psi^{n-1}(X)$ s.t.

$y \in \psi(z_n)$

$\{z_n\} \subseteq X$ compact

$\Rightarrow \exists \{z_{n_k}\}$ s.t. $z_{n_k} \to z \in X$

$\Rightarrow y \in \psi(z)$ $\left((z_{n_k}, y) \in \text{gr } \psi \ \forall k \right)$

Show $z \in C$. : If not, $z \notin C$

$C = \bigcap_{n \in \mathbb{N}} \psi^n(X) \Rightarrow \exists N$ s.t. $z \notin \psi^N(X)$

$\Rightarrow \exists \varepsilon > 0$ s.t. $B_\varepsilon(z) \cap \psi^N(X) = \emptyset$