Econ 204 2021
Lecture 2

Outline

1. Cardinality (cont.)
2. Algebraic Structures: Fields and Vector Spaces
3. Axioms for \( \mathbb{R} \)
4. Sup, Inf, and the Supremum Property
5. Intermediate Value Theorem

Announcements
- PSL due Friday
  1 pm in bCourses
- marked slides posted on class website after lectures
Cardinality

Definition 5. Two sets $A, B$ are numerically equivalent (or have the same cardinality) if there is a bijection $f : A \to B$, that is, a function $f : A \to B$ that is 1-1 ($a \neq a' \Rightarrow f(a) \neq f(a')$), and onto ($\forall b \in B \ \exists a \in A \text{ s.t. } f(a) = b$).

Example: $A = \{2, 4, 6, \ldots, 50\}$ is numerically equivalent to the set $\{1, 2, \ldots, 25\}$ under the function $f(n) = 2n$.

$B = \{1, 4, 9, 16, 25, 36, 49 \ldots\} = \{n^2 : n \in \mathbb{N}\}$ is numerically equivalent to $\mathbb{N}$. 
Cardinality

A set is either finite or infinite. A set is \textit{finite} if it is numerically equivalent to \( \{1, \ldots, n\} \) for some \( n \). A set that is not finite is \textit{infinite}.

In particular, \( A = \{2, 4, 6, \ldots, 50\} \) is finite, \( B = \{1, 4, 9, 16, 25, 36, 49 \ldots\} \) is infinite.

A set is \textit{countable} if it is numerically equivalent to the set of natural numbers \( \mathbb{N} = \{1, 2, 3, \ldots\} \). An infinite set that is not countable is called \textit{uncountable}. 
Cardinality

Example: The set of integers \( \mathbb{Z} \) is countable.

\[
\mathbb{Z} = \{0, 1, -1, 2, -2, \ldots\}
\]

Define \( f : \mathbb{N} \rightarrow \mathbb{Z} \) by

\[
\begin{align*}
    f(1) &= 0 \\
    f(2) &= 1 \\
    f(3) &= -1 \\
    \vdots
    \end{align*}
\]

\[
f(n) = (-1)^n \left\lfloor \frac{n}{2} \right\rfloor
\]

where \( \lfloor x \rfloor \) is the greatest integer less than or equal to \( x \). It is straightforward to verify that \( f \) is one-to-one and onto.
Cardinality

Theorem 5. The set of rational numbers $\mathbb{Q}$ is countable.

“Picture Proof”:

$$\mathbb{Q} = \left\{ \frac{m}{n} : m, n \in \mathbb{Z}, n \neq 0 \right\}$$

$$= \left\{ \frac{m}{n} : m \in \mathbb{Z}, n \in \mathbb{N} \right\}$$
Go back and forth on upward-sloping diagonals, omitting the
repeats:

\[ f(1) = 0 \]
\[ f(2) = 1 \]
\[ f(3) = \frac{1}{2} \]
\[ f(4) = -1 \]
\[ \vdots \]

\[ f : \mathbb{N} \rightarrow \mathbb{Q}, \] \( f \) is one-to-one and onto.
Cardinality (cont.)

**Notation:** Given a set $A$, $2^A$ is the set of all subsets of $A$. This is the “power set” of $A$, also denoted $P(A)$.

Important example of an uncountable set:

**Theorem 1** (Cantor). $2^\mathbb{N}$, the set of all subsets of $\mathbb{N}$, is not countable.

*Proof.* Suppose $2^\mathbb{N}$ is countable. Then there is a bijection $f : \mathbb{N} \to 2^\mathbb{N}$. Let $A_m = f(m)$. We create an infinite matrix, whose
\((m, n)^{th}\) entry is 1 if \(n \in A_m\), 0 otherwise:

<table>
<thead>
<tr>
<th></th>
<th>(N)</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>\ldots</th>
</tr>
</thead>
<tbody>
<tr>
<td>(A_1) = (\emptyset)</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>\ldots</td>
</tr>
<tr>
<td>(A_2) = {1}</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>\ldots</td>
</tr>
<tr>
<td>(2^N) (A_3) = {1, 2, 3}</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>\ldots</td>
</tr>
<tr>
<td>(A_4) = (\mathbb{N})</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>\ldots</td>
</tr>
<tr>
<td>(A_5) = (2^\mathbb{N})</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>\ldots</td>
</tr>
<tr>
<td>\vdots</td>
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<td>\vdots</td>
<td>\vdots</td>
<td>\vdots</td>
<td>\vdots</td>
<td>\ldots</td>
</tr>
</tbody>
</table>

Now, on the main diagonal, change all the 0s to 1s and vice
versa:

<table>
<thead>
<tr>
<th></th>
<th>$N$</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>...</th>
</tr>
</thead>
<tbody>
<tr>
<td>$f(1)$</td>
<td>$A_1 = \emptyset$</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>...</td>
</tr>
<tr>
<td>$f(2)$</td>
<td>$A_2 = {1}$</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>...</td>
</tr>
<tr>
<td>$2^N$</td>
<td>$A_3 = {1, 2, 3}$</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>...</td>
</tr>
<tr>
<td></td>
<td>$A_4 = N$</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>...</td>
</tr>
<tr>
<td></td>
<td>$A_5 = 2N$</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>...</td>
</tr>
<tr>
<td></td>
<td>$\vdots$</td>
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<td>$\vdots$</td>
<td>$\vdots$</td>
<td>$\vdots$</td>
<td>$\vdots$</td>
</tr>
</tbody>
</table>
Let

\[ t_{mn} = \begin{cases} 
1 & \text{if } n \in A_m \\
0 & \text{if } n \notin A_m 
\end{cases} \]

Let \( A = \{m \in \mathbb{N} : t_{mm} = 0\} \).

\[
m \in A \iff t_{mm} = 0 \\
\iff m \notin A_m \\
1 \in A \iff 1 \notin A_1 \text{ so } A \neq A_1 \\
2 \in A \iff 2 \notin A_2 \text{ so } A \neq A_2 \\
\vdots \\
m \in A \iff m \notin A_m \text{ so } A \neq A_m \quad \forall n \in \mathbb{N}
\]

Therefore, \( A \neq f(m) \) for any \( m \), so \( f \) is not onto, contradiction. \( \Box \)
Some Additional Facts About Cardinality

Recall we let \(|A|\) denote the cardinality of a set \(A\).

- if \(A\) is numerically equivalent to \(\{1, \ldots, n\}\) for some \(n \in \mathbb{N}\), then \(|A| = n\).

- \(A\) and \(B\) are numerically equivalent if and only if \(|A| = |B|\)

- if \(|A| = n\) and \(A\) is a proper subset of \(B\) (that is, \(A \subseteq B\) and \(A \neq B\)) then \(|A| < |B|\)
• if $A$ is countable and $B$ is uncountable, then
  $$n < |A| < |B| \quad \forall n \in \mathbb{N}$$

• if $A \subseteq B$ then $|A| \leq |B|$:

• if $r : A \rightarrow B$ is 1-1, then $|A| \leq |B|$:

• if $B$ is countable and $A \subseteq B$, then $A$ is at most countable, that is, $A$ is either empty, finite, or countable

• if $r : A \rightarrow B$ is 1-1 and $B$ is countable, then $A$ is at most countable
Algebraic Structures: Fields

**Definition 1.** A field \( \mathcal{F} = (F, +, \cdot) \) is a 3-tuple consisting of a set \( F \) and two binary operations \(+, \cdot : F \times F \to F\) such that

1. **Associativity of \(+\):**
   \[
   \forall \alpha, \beta, \gamma \in F, \quad (\alpha + \beta) + \gamma = \alpha + (\beta + \gamma)
   \]

2. **Commutativity of \(+\):**
   \[
   \forall \alpha, \beta \in F, \quad \alpha + \beta = \beta + \alpha
   \]

3. **Existence of additive identity:**
   \[
   \exists! 0 \in F \text{ s.t. } \forall \alpha \in F, \quad \alpha + 0 = 0 + \alpha = \alpha
   \]
4. Existence of additive inverse:
\[ \forall \alpha \in F \; \exists!(-\alpha) \in F \; s.t. \; \alpha + (-\alpha) = (-\alpha) + \alpha = 0 \]
Define \( \alpha - \beta = \alpha + (-\beta) \)

5. Associativity of \( \cdot \):
\[ \forall \alpha, \beta, \gamma \in F, \; (\alpha \cdot \beta) \cdot \gamma = \alpha \cdot (\beta \cdot \gamma) \]

6. Commutativity of \( \cdot \):
\[ \forall \alpha, \beta \in F, \; \alpha \cdot \beta = \beta \cdot \alpha \]

7. Existence of multiplicative identity:
\[ \exists!1 \in F \; s.t. \; 1 \neq 0 \; and \; \forall \alpha \in F, \; \alpha \cdot 1 = 1 \cdot \alpha = \alpha \]
8. Existence of multiplicative inverse:

\[ \forall \alpha \in F \text{ s.t. } \alpha \neq 0 \exists! \alpha^{-1} \in F \text{ s.t. } \alpha \cdot \alpha^{-1} = \alpha^{-1} \cdot \alpha = 1 \]

Define \( \frac{\alpha}{\beta} = \alpha \beta^{-1} \). \( \beta \neq 0 \)

9. Distributivity of multiplication over addition:

\[ \forall \alpha, \beta, \gamma \in F, \ \alpha \cdot (\beta + \gamma) = \alpha \cdot \beta + \alpha \cdot \gamma \]
Fields

Examples:

- \( R \) \( \text{standard } +, \cdot \)
  - \( \mathbb{C} \) \( \text{complex numbers} \)
    - \( \mathbb{C} = \{ x + iy : x, y \in \mathbb{R} \} \). \( i^2 = -1 \), so
      \[
      (x+iy)(w+iz) = xw+ixz+iwy+i^2yz = (xw-yz)+i(xz+wy)
      \]
  - \( \mathbb{Q} : \mathbb{Q} \subset \mathbb{R}, \mathbb{Q} \neq \mathbb{R} \). \( \mathbb{Q} \) is closed under \( +, \cdot \), taking additive and multiplicative inverses; the field axioms are inherited from the field axioms on \( \mathbb{R} \), so \( \mathbb{Q} \) is a field.
• $\mathbb{N}$ is not a field: no additive identity. $m + n \neq m$ for $n \neq 0$

• $\mathbb{Z}$ is not a field; no multiplicative inverse for 2. $\nexists z \in \mathbb{Z}$ s.t. $z^2 = 1$

• $\mathbb{Q}(\sqrt{2})$, the smallest field containing $\mathbb{Q} \cup \{\sqrt{2}\}$. Take $\mathbb{Q}$, add $\sqrt{2}$, and close up under $+$, $\cdot$, taking additive and multiplicative inverses. One can show

$$\mathbb{Q}(\sqrt{2}) = \{ q + r\sqrt{2} : q, r \in \mathbb{Q} \}$$

For example,

$$\left(q + r\sqrt{2}\right)^{-1} = \frac{q}{q^2 - 2r^2} - \frac{r}{q^2 - 2r^2}\sqrt{2}$$
A finite field: $F_2 = (\{0, 1\}, +, \cdot)$ where we define

\[
\begin{align*}
0 + 0 &= 0 & 0 \cdot 0 &= 0 \\
0 + 1 &= 1 + 0 &= 1 & 0 \cdot 1 &= 1 \cdot 0 &= 0 \\
1 + 1 &= 0 & 1 \cdot 1 &= 1
\end{align*}
\]

("Arithmetic mod 2") \(\sim\Rightarrow\ 1 = -1\)
Vector Spaces

**Definition 2.** A vector space is a 4-tuple \((V, F, +, \cdot)\) where \(V\) is a set of elements, called vectors, \(F\) is a field, \(+\) is a binary operation on \(V\) called vector addition, and \(\cdot : F \times V \rightarrow V\) is called scalar multiplication, satisfying

1. **Associativity of \(+\):**

   \[\forall x, y, z \in V, \ (x + y) + z = x + (y + z)\]

2. **Commutativity of \(+\):**

   \[\forall x, y \in V, \ x + y = y + x\]
3. Existence of vector additive identity:

\[ \exists! 0 \in V \text{ s.t. } \forall x \in V, \ x + 0 = 0 + x = x \]

4. Existence of vector additive inverse:

\[ \forall x \in V \ \exists! (-x) \in V \text{ s.t. } x + (-x) = (-x) + x = 0 \]

Define \( x - y \) to be \( x + (-y) \).

5. Distributivity of scalar multiplication over vector addition:

\[ \forall \alpha \in F, x, y \in V, \ \alpha \cdot (x + y) = \alpha \cdot x + \alpha \cdot y \]

6. Distributivity of scalar multiplication over scalar addition:

\[ \forall \alpha, \beta \in F, x \in V \ \ (\alpha + \beta) \cdot x = \alpha \cdot x + \beta \cdot x \]
7. Associativity of $\cdot$:

$$\forall \alpha, \beta \in F, x \in V \quad (\alpha \cdot \beta) \cdot x = \alpha \cdot (\beta \cdot x)$$

8. Multiplicative identity:

$$\forall x \in V \quad 1 \cdot x = x$$

(Note that 1 is the multiplicative identity in $F$; $1 \not\in V$)

"$V$ is a vector space over $F$"

or "$V$ over $F"
Vector Spaces

Examples:

1. $\mathbb{R}^n$ over $\mathbb{R}$.

2. $\mathbb{R}$ is a vector space over $\mathbb{Q}$:

   (scalar multiplication) $q \cdot r = qr$ (product in $\mathbb{R}$)

   $\mathbb{R}$ is not finite-dimensional over $\mathbb{Q}$, i.e. $\mathbb{R}$ is not $\mathbb{Q}^n$ for any $n \in \mathbb{N}$.

3. $\mathbb{R}$ is a vector space over $\mathbb{R}$. 
4. \( \mathbb{Q}(\sqrt{2}) \) is a vector space over \( \mathbb{Q} \). As a vector space, it is \( \mathbb{Q}^2 \); as a field, you need to take the funny field multiplication. i.e. \((q, r)\) versus \(q + r\sqrt{2}\)

5. \( \mathbb{Q}(\sqrt[3]{2}) \), as a vector space over \( \mathbb{Q} \), is \( \mathbb{Q}^3 \).

6. \((F_2)^n\) is a finite vector space over \( F_2 \).

7. \( C([0, 1]) \), the space of all continuous real-valued functions on \([0, 1]\), is a vector space over \( \mathbb{R} \).

- Vector addition: \( f, g \in C([0, 1]) \)

\[ (f + g)(t) = f(t) + g(t) \quad \forall t \in [0, 1] \]
Note we define the function $f + g$ by specifying what value it takes for each $t \in [0, 1]$.

- scalar multiplication: $\alpha \in \mathbb{R}$, $f \in C([0,1])$
  \[(\alpha f)(t) = \alpha(f(t)) \quad \forall t \in [0,1]\]

- vector additive identity: 0 is the function which is identically zero: $0(t) = 0$ for all $t \in [0,1]$.

- vector additive inverse:
  \[(-f)(t) = -(f(t)) \quad \forall t \in [0,1]\]
Axioms for \( \mathbb{R} \)

1. \( \mathbb{R} \) is a field with the usual operations \(+\), \(\cdot\), additive identity 0, and multiplicative identity 1.

2. **Order Axiom:** There is a complete ordering \(\leq\), i.e. \(\leq\) is reflexive, transitive, antisymmetric \((\alpha \leq \beta, \beta \leq \alpha \Rightarrow \alpha = \beta)\) with the property that

\[
\forall \alpha, \beta \in \mathbb{R} \text{ either } \alpha \leq \beta \text{ or } \beta \leq \alpha
\]

The order is compatible with \(+\) and \(\cdot\), i.e.

\[
\forall \alpha, \beta, \gamma \in \mathbb{R} \left\{ \begin{array}{l}
\alpha \leq \beta \Rightarrow \alpha + \gamma \leq \beta + \gamma \\
\alpha \leq \beta, 0 \leq \gamma \Rightarrow \alpha \gamma \leq \beta \gamma
\end{array} \right.
\]

\(\alpha \geq \beta\) means \(\beta \leq \alpha\). \(\alpha < \beta\) means \(\alpha \leq \beta\) and \(\alpha \neq \beta\).
Completeness Axiom

3. **Completeness Axiom:** Suppose $L, H \subseteq \mathbb{R}$, $L \neq \emptyset \neq H$ satisfy

$$\ell \leq h \ \forall \ell \in L, h \in H$$

Then

$$\exists \alpha \in \mathbb{R} \text{ s.t. } \ell \leq \alpha \leq h \ \forall \ell \in L, h \in H$$

The Completeness Axiom differentiates $\mathbb{R}$ from $\mathbb{Q}$: $\mathbb{Q}$ satisfies all the axioms for $\mathbb{R}$ except the Completeness Axiom.
Sups, Infs, and the Supremum Property

Definition 3. Suppose $X \subseteq \mathbb{R}$. We say $u \in \mathbb{R}$ is an upper bound for $X$ if

$$x \leq u \ \forall x \in X$$

and $\ell \in \mathbb{R}$ is a lower bound for $X$ if

$$\ell \leq x \ \forall x \in X$$

$X$ is bounded above if there is an upper bound for $X$, and bounded below if there is a lower bound for $X$. 
Definition 4. Suppose \( X \) is bounded above. The supremum of \( X \), written \( \sup X \), is the least upper bound for \( X \), i.e. \( \sup X \in \mathbb{R} \) satisfies

\[
\sup X \geq x \quad \forall x \in X \quad (\text{sup } X \text{ is an upper bound})
\]

\[
\forall y < \sup X \ \exists x \in X \text{ s.t. } x > y \quad (\text{there is no smaller upper bound})
\]

Analogously, suppose \( X \) is bounded below. The infimum of \( X \), written \( \inf X \), is the greatest lower bound for \( X \), i.e. \( \inf X \) satisfies

\[
\inf X \leq x \quad \forall x \in X \quad (\text{inf } X \text{ is a lower bound})
\]

\[
\forall y > \inf X \ \exists x \in X \text{ s.t. } x < y \quad (\text{there is no greater lower bound})
\]

If \( X \) is not bounded above, write \( \sup X = \infty \). If \( X \) is not bounded below, write \( \inf X = -\infty \). Convention: \( \sup \emptyset = -\infty \), \( \inf \emptyset = +\infty \).
The Supremum Property

**The Supremum Property:** Every nonempty set of real numbers that is bounded above has a supremum, which is a real number. Every nonempty set of real numbers that is bounded below has an infimum, which is a real number.

**Note:** \( \sup X \) need not be an element of \( X \). For example, \( \sup (0, 1) = 1 \notin (0, 1) \).
The Supremum Property

Theorem 2 (Theorem 6.8, plus ...). The Supremum Property and the Completeness Axiom are equivalent.

Proof. Assume the Completeness Axiom. Let $X \subseteq \mathbb{R}$ be a nonempty set that is bounded above. Let $U$ be the set of all upper bounds for $X$. Since $X$ is bounded above, $U \neq \emptyset$. If $x \in X$ and $u \in U$, $x \leq u$ since $u$ is an upper bound for $X$. So

$$x \leq u \ \forall x \in X, u \in U$$

By the Completeness Axiom,

$$\exists \alpha \in \mathbb{R} \text{ s.t. } x \leq \alpha \leq u \ \forall x \in X, u \in U$$

$\alpha$ is an upper bound for $X$, and it is less than or equal to every other upper bound for $X$, so it is the least upper bound for $X$,
so \( \sup X = \alpha \in \mathbb{R} \). The case in which \( X \) is bounded below is similar. Thus, the Supremum Property holds.

Conversely, assume the Supremum Property. Suppose \( L, H \subseteq \mathbb{R}, L \neq \emptyset \neq H \), and

\[
\ell \leq h \ \forall \ell \in L, h \in H
\]

Since \( L \neq \emptyset \) and \( L \) is bounded above (by any element of \( H \)), \( \alpha = \sup L \) exists and is real. By the definition of supremum, \( \alpha \) is an upper bound for \( L \), so

\[
\ell \leq \alpha \ \forall \ell \in L
\]

Suppose \( h \in H \). Then \( h \) is an upper bound for \( L \), so by the definition of supremum, \( \alpha \leq h \). Therefore, we have shown that

\[
\ell \leq \alpha \leq h \ \forall \ell \in L, h \in H
\]

so the Completeness Axiom holds. \( \square \)
Archimedean Property

**Theorem 3** (Archimedean Property, Theorem 6.10 + ...).

\[ \forall x, y \in \mathbb{R}, y > 0 \ \exists n \in \mathbb{N} \text{ s.t. } ny = (y + \cdots + y) > x \]

**Proof.** Exercise. This is a nice exercise in proof by contradiction, using the Supremum Property. \(\square\)
Intermediate Value Theorem

**Theorem 4** (Intermediate Value Theorem). *Suppose $f : [a, b] \to \mathbb{R}$ is continuous, and $f(a) < d < f(b)$. Then there exists $c \in (a, b)$ such that $f(c) = d$.*

*Proof.* Later, we will give a slick proof. Here, we give a bare-hands proof using the Supremum Property. Let

$$B = \{ x \in [a, b] : f(x) < d \}$$

$a \in B$, so $B \neq \emptyset$; $B \subseteq [a, b]$, so $B$ is bounded above. By the Supremum Property, $\sup B$ exists and is real so let $c = \sup B$. Since $a \in B$, $c \geq a$. $B \subseteq [a, b]$, so $c \leq b$. Therefore, $c \in [a, b]$. 

15
$f(a) < d < f(b)$

$B = \{ x \in [a, b] : f(x) < d \}$

$c = \sup B$

Claim: $f(c) = d$
We claim that $f(c) = d$. If not, suppose $f(c) < d$. Then since $f(b) > d$, $c \neq b$, so $c < b$. Let $\varepsilon = \frac{d-f(c)}{2} > 0$. Since $f$ is continuous at $c$, there exists $\delta > 0$ such that

$$|x - c| < \delta \Rightarrow |f(x) - f(c)| < \varepsilon$$

$$\Rightarrow f(x) < f(c) + \varepsilon$$

$$= f(c) + \frac{d-f(c)}{2}$$

$$= \frac{f(c)+d}{2}$$

$$< \frac{d+d}{2}$$

$$= d$$

so $(c, c + \delta) \subseteq B$, so $c \neq \sup B$, contradiction.
$f(c) < s \Rightarrow \exists \delta > 0 \text{ s.t. for } x \in (c - \delta, c + \delta), f(x) < d$

$\Rightarrow c \neq \sup B$
Suppose \( f(c) > d \). Then since \( f(a) < d \), \( a \neq c \), so \( c > a \). Let 
\[
\varepsilon = \frac{\frac{f(c) - d}{2}}{2} > 0.
\]
Since \( f \) is continuous at \( c \), there exists \( \delta > 0 \) such that
\[
|x - c| < \delta \quad \Rightarrow \quad |f(x) - f(c)| < \varepsilon
\]
\[
\Rightarrow \quad f(x) > f(c) - \varepsilon
\]
\[
= f(c) - \frac{f(c) - d}{2}
\]
\[
= \frac{f(c) + d}{2}
\]
\[
> \frac{d + d}{2}
\]
\[
= d
\]
so \((c - \delta, c + \delta) \cap B = \emptyset\). So either there exists \( x \in B \) with \( x \geq c + \delta \)
(in which case \( c \) is not an upper bound for \( B \)) or \( c - \delta \) is an upper bound for \( B \)
(in which case \( c \) is not the least upper bound for \( B \)); in either case, \( c \neq \sup B \), contradiction.
\[ f(c) > d \implies \exists \delta > 0 \text{ s.t. } f(x) > d, \forall x \in (c-\delta, c+\delta) \]

\[(c-\delta, c+\delta) \cap B = \emptyset \implies \text{ either } \exists y \in [c+\delta, b] \cap B \]
\[\text{ or } B \subset [a, c-\delta] \]

in either case, \( c = \sup B \)
Since $f(c) \not< d$, $f(c) \not> d$, and the order is complete, $f(c) = d$. Since $f(a) < d$ and $f(b) > d$, $a \neq c \neq b$, so $c \in (a, b)$. 
\[\square\]
\[ x = 3 \cdot 1.2^3 \]

\[ x/R = \{ [17], [23] \} \]

---

\[ f : A \rightarrow B \]

\[ \forall a \in A \quad \exists b \in B \text{ s.t. } f(a) = b \]

\[ f \text{ onto } \Rightarrow \exists b \in B \quad \exists a \in A \text{ s.t. } f(a) = b \]

\[ f \text{ 1-1 } \Rightarrow a \neq a' \Rightarrow f(a) \neq f(a') \]
Corollary 1. There exists $x \in \mathbb{R}$ such that $x^2 = 2$.

Proof. Let $f(x) = x^2$, for $x \in [0, 2]$. $f$ is continuous (Why?). $f(0) = 0 < 2$ and $f(2) = 4 > 2$, so by the Intermediate Value Theorem, there exists $c \in (0, 2)$ such that $f(c) = 2$, i.e. such that $c^2 = 2$. \qed