Econ 204 2021

Lecture 2

Outline

1. Cardinality (cont.)
2. Algebraic Structures: Fields and Vector Spaces
3. Axioms for $\mathbf{R}$
4. Sup, Inf, and the Supremum Property
5. Intermediate Value Theorem

Cardinality (cont.)

Notation: Given a set A, 2^A is the set of all subsets of A. This is the "power set" of A, also denoted $P(A)$.

Important example of an uncountable set:

Theorem 1 (Cantor)**.** $2^{\mathbf{N}}$, *the set of all subsets of* $\mathbf{N}$, *is* not *countable.*

Proof. Suppose $2^{\mathbf{N}}$ is countable. Then there is a bijection $f : \mathbf{N} \to 2^{\mathbf{N}}$. Let $A_m = f(m)$. We create an infinite matrix, whose

$(m,n)^{th}$ entry is 1 if $n \in A_m$, 0 otherwise:

				$\mathbf{N}$					
				1	2	3	4	5	$\cdots$
	A_1	=	$\emptyset$	$\mathbf{0}$	0	0	0	0	$\cdots$
	A_2	=	$\{1\}$	1	$\mathbf{0}$	0	0	0	$\cdots$
$2^{\mathbf{N}}$	A_3	=	$\{1,2,3\}$	1	1	$\mathbf{1}$	0	0	$\cdots$
	A_4	=	$\mathbf{N}$	1	1	1	$\mathbf{1}$	1	$\cdots$
	A_5	=	$2\mathbf{N}$	0	1	0	1	$\mathbf{0}$	$\cdots$
		$\vdots$		$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

Now, on the main diagonal, change all the 0s to 1s and vice versa:

						$\mathbf{N}$			
				1	2	3	4	5	$\cdots$
	A_1	$=$	$\emptyset$	$\mathbf{1}$	0	0	0	0	$\cdots$
	A_2	$=$	$\{1\}$	1	$\mathbf{1}$	0	0	0	$\cdots$
$2^{\mathbf{N}}$	A_3	$=$	$\{1, 2, 3\}$	1	1	$\mathbf{0}$	0	0	$\cdots$
	A_4	$=$	$\mathbf{N}$	1	1	1	$\mathbf{0}$	1	$\cdots$
	A_5	$=$	$2\mathbf{N}$	0	1	0	1	$\mathbf{1}$	$\cdots$
		$\vdots$		$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

Let

$$t_{mn} = \begin{cases} 1 & \text{if } n \in A_m \\ 0 & \text{if } n \notin A_m \end{cases}$$

Let $A = \{m \in \mathbf{N} : t_{mm} = 0\}$.

$$\begin{aligned} m \in A &\Leftrightarrow t_{mm} = 0 \\ &\Leftrightarrow m \notin A_m \\ 1 \in A &\Leftrightarrow 1 \notin A_1 \text{ so } A \neq A_1 \\ 2 \in A &\Leftrightarrow 2 \notin A_2 \text{ so } A \neq A_2 \\ &\vdots \\ m \in A &\Leftrightarrow m \notin A_m \text{ so } A \neq A_m \end{aligned}$$

Therefore, $A \neq f(m)$ for any m, so f is not onto, contradiction.

□

Some Additional Facts About Cardinality

Recall we let $|A|$ denote the cardinality of a set A.

- if A is numerically equivalent to $\{1, \ldots, n\}$ for some $n \in \mathbf{N}$, then $|A| = n$.

- A and B are numerically equivalent if and only if $|A| = |B|$

- if $|A| = n$ and A is a proper subset of B (that is, $A \subseteq B$ and $A \neq B$) then $|A| < |B|$

- if A is countable and B is uncountable, then

$$n < |A| < |B| \quad \forall n \in \mathbf{N}$$

- if $A \subseteq B$ then $|A| \leq |B|$

- if $r : A \to B$ is 1-1, then $|A| \leq |B|$

- if B is countable and $A \subseteq B$, then A is at most countable, that is, A is either empty, finite, or countable

- if $r : A \to B$ is 1-1 and B is countable, then A is at most countable

Algebraic Structures: Fields

Definition 1. *A* field $\mathcal{F} = (F, +, \cdot)$ *is a 3-tuple consisting of a set* F *and two binary operations* $+, \cdot : F \times F \to F$ *such that*

1. *Associativity of* $+$*:*

$$\forall \alpha, \beta, \gamma \in F,\ (\alpha + \beta) + \gamma = \alpha + (\beta + \gamma)$$

2. *Commutativity of* $+$*:*

$$\forall \alpha, \beta \in F,\ \alpha + \beta = \beta + \alpha$$

3. *Existence of additive identity:*

$$\exists! 0 \in F \text{ s.t. } \forall \alpha \in F,\ \alpha + 0 = 0 + \alpha = \alpha$$

4. *Existence of additive inverse:*

$$\forall \alpha \in F \ \exists!(-\alpha) \in F \ \textit{s.t.} \ \alpha + (-\alpha) = (-\alpha) + \alpha = 0$$

Define $\alpha - \beta = \alpha + (-\beta)$

5. *Associativity of* $\cdot$ *:*

$$\forall \alpha, \beta, \gamma \in F, \ (\alpha \cdot \beta) \cdot \gamma = \alpha \cdot (\beta \cdot \gamma)$$

6. *Commutativity of* $\cdot$ *:*

$$\forall \alpha, \beta \in F, \ \alpha \cdot \beta = \beta \cdot \alpha$$

7. *Existence of multiplicative identity:*

$$\exists! 1 \in F \ \textit{s.t.} \ 1 \neq 0 \ \textit{and} \ \forall \alpha \in F, \ \alpha \cdot 1 = 1 \cdot \alpha = \alpha$$

8. *Existence of multiplicative inverse:*

$$\forall \alpha \in F \text{ s.t. } \alpha \neq 0 \ \exists! \alpha^{-1} \in F \text{ s.t. } \alpha \cdot \alpha^{-1} = \alpha^{-1} \cdot \alpha = 1$$

Define $\frac{\alpha}{\beta} = \alpha\beta^{-1}$.

9. *Distributivity of multiplication over addition:*

$$\forall \alpha, \beta, \gamma \in F, \ \alpha \cdot (\beta + \gamma) = \alpha \cdot \beta + \alpha \cdot \gamma$$

Fields

Examples:

- $\mathbf{R}$

- $\mathbf{C} = \{x + iy : x, y \in \mathbf{R}\}$. $i^2 = -1$, so

  $(x+iy)(w+iz) = xw+ixz+iwy+i^2yz = (xw-yz)+i(xz+wy)$

- $\mathbf{Q}$: $\mathbf{Q} \subset \mathbf{R}$, $\mathbf{Q} \neq \mathbf{R}$. $\mathbf{Q}$ is closed under $+, \cdot$, taking additive and multiplicative inverses; the field axioms are inherited from the field axioms on $\mathbf{R}$, so $\mathbf{Q}$ is a field.

- $\mathbf{N}$ is not a field: no additive identity.

- $\mathbf{Z}$ is not a field; no multiplicative inverse for 2.

- $\mathbf{Q}(\sqrt{2})$, the smallest field containing $\mathbf{Q} \cup \{\sqrt{2}\}$. Take $\mathbf{Q}$, add $\sqrt{2}$, and close up under $+$, $\cdot$, taking additive and multiplicative inverses. One can show

$$\mathbf{Q}(\sqrt{2}) = \{q + r\sqrt{2} : q, r \in \mathbf{Q}\}$$

For example,

$$\left(q + r\sqrt{2}\right)^{-1} = \frac{q}{q^2 - 2r^2} - \frac{r}{q^2 - 2r^2}\sqrt{2}$$

- A *finite field*: $F_2 = (\{0,1\},+,\cdot)$ where

$$\begin{array}{rcccl} & & 0+0 & = & 0 \\ 0+1 & = & 1+0 & = & 1 \\ & & 1+1 & = & 0 \end{array} \qquad \begin{array}{rcccl} & & 0\cdot 0 & = & 0 \\ 0\cdot 1 & = & 1\cdot 0 & = & 0 \\ & & 1\cdot 1 & = & 1 \end{array}$$

(*"Arithmetic mod 2"*)

Vector Spaces

Definition 2. *A* vector space *is a 4-tuple* $(V, F, +, \cdot)$ *where* V *is a set of elements, called* vectors, F *is a field,* $+$ *is a binary operation on* V *called vector addition, and* $\cdot : F \times V \to V$ *is called scalar multiplication, satisfying*

1. *Associativity of* $+$*:*

$$\forall x, y, z \in V,\ (x + y) + z = x + (y + z)$$

2. *Commutativity of* $+$*:*

$$\forall x, y \in V,\ x + y = y + x$$

3. *Existence of vector additive identity:*

$$\exists! 0 \in V \text{ s.t. } \forall x \in V,\ x + 0 = 0 + x = x$$

4. *Existence of vector additive inverse:*

$$\forall x \in V\ \exists!(-x) \in V \text{ s.t. } x + (-x) = (-x) + x = 0$$

Define $x - y$ *to be* $x + (-y)$.

5. *Distributivity of scalar multiplication over vector addition:*

$$\forall \alpha \in F, x, y \in V,\ \alpha \cdot (x + y) = \alpha \cdot x + \alpha \cdot y$$

6. *Distributivity of scalar multiplication over scalar addition:*

$$\forall \alpha, \beta \in F, x \in V \quad (\alpha + \beta) \cdot x = \alpha \cdot x + \beta \cdot x$$

7. *Associativity of $\cdot$:*

$$\forall \alpha, \beta \in F, x \in V \quad (\alpha \cdot \beta) \cdot x = \alpha \cdot (\beta \cdot x)$$

8. *Multiplicative identity:*

$$\forall x \in V \quad 1 \cdot x = x$$

(Note that 1 *is the multiplicative identity in* F*;* $1 \notin V$ *)*

Vector Spaces

Examples:

1. $\mathbf{R}^n$ over $\mathbf{R}$.

2. $\mathbf{R}$ is a vector space over $\mathbf{Q}$:

   (scalar multiplication) $q \cdot r = qr$ (product in $\mathbf{R}$)

   $\mathbf{R}$ is not finite-dimensional over $\mathbf{Q}$, i.e. $\mathbf{R}$ is not $\mathbf{Q}^n$ for any $n \in \mathbf{N}$.

3. $\mathbf{R}$ is a vector space over $\mathbf{R}$.

4. $\mathbf{Q}(\sqrt{2})$ is a vector space over $\mathbf{Q}$. As a vector space, it is $\mathbf{Q}^2$; as a field, you need to take the funny field multiplication.

5. $\mathbf{Q}(\sqrt[3]{2})$, as a vector space over $\mathbf{Q}$, is $\mathbf{Q}^3$.

6. $(F_2)^n$ is a *finite* vector space over F_2.

7. $C([0,1])$, the space of all continuous real-valued functions on $[0,1]$, is a vector space over $\mathbf{R}$.

   - vector addition:

$$(f+g)(t) = f(t) + g(t)$$

Note we define the function $f+g$ by specifying what value it takes for each $t \in [0, 1]$.

- scalar multiplication:

$$(\alpha f)(t) = \alpha(f(t))$$

- vector additive identity: 0 is the function which is identically zero: $0(t) = 0$ for all $t \in [0, 1]$.

- vector additive inverse:

$$(-f)(t) = -(f(t))$$

Axioms for $\mathbf{R}$

1. $\mathbf{R}$ is a field with the usual operations $+$, $\cdot$, additive identity 0, and multiplicative identity 1.

2. **Order Axiom:** There is a complete ordering $\leq$, i.e. $\leq$ is reflexive, transitive, antisymmetric ($\alpha \leq \beta, \beta \leq \alpha \Rightarrow \alpha = \beta$) with the property that

$$\forall \alpha, \beta \in \mathbf{R} \text{ either } \alpha \leq \beta \text{ or } \beta \leq \alpha$$

The order is compatible with $+$ and $\cdot$, i.e.

$$\forall \alpha, \beta, \gamma \in \mathbf{R} \begin{cases} \alpha \leq \beta & \Rightarrow \quad \alpha + \gamma \leq \beta + \gamma \\ \alpha \leq \beta, 0 \leq \gamma & \Rightarrow \quad \alpha\gamma \leq \beta\gamma \end{cases}$$

$\alpha \geq \beta$ means $\beta \leq \alpha$. $\alpha < \beta$ means $\alpha \leq \beta$ and $\alpha \neq \beta$.

Completeness Axiom

3. **Completeness Axiom:** Suppose $L, H \subseteq \mathbf{R}$, $L \neq \emptyset \neq H$ satisfy

$$\ell \leq h \quad \forall \ell \in L, h \in H$$

Then

$$\exists \alpha \in \mathbf{R} \text{ s.t. } \ell \leq \alpha \leq h \quad \forall \ell \in L, h \in H$$

$$\begin{array}{ccc} & \alpha & \\ L & \downarrow & H \\ ----) & \cdot & (---- \end{array}$$

The Completeness Axiom differentiates $\mathbf{R}$ from $\mathbf{Q}$: $\mathbf{Q}$ satisfies all the axioms for $\mathbf{R}$ except the Completeness Axiom.

Sups, Infs, and the Supremum Property

Definition 3. *Suppose* $X \subseteq \mathbf{R}$. *We say* u *is an* upper bound *for* X *if*

$$x \leq u \ \forall x \in X$$

and ℓ *is a* lower bound *for* X *if*

$$\ell \leq x \ \forall x \in X$$

X *is* bounded above *if there is an upper bound for* X, *and* bounded below *if there is a lower bound for* X.

Definition 4. *Suppose X is bounded above. The* supremum *of X, written* $\sup X$, *is the least upper bound for X, i.e.* $\sup X$ *satisfies*

$$\sup X \geq x \ \ \forall x \in X \text{ (}\sup X \textit{ is an upper bound)}$$

$$\forall y < \sup X \ \exists x \in X \textit{ s.t. } x > y \textit{ (there is no smaller upper bound)}$$

Analogously, suppose X is bounded below. The infimum *of X, written* $\inf X$, *is the greatest lower bound for X, i.e.* $\inf X$ *satisfies*

$$\inf X \leq x \ \ \forall x \in X \text{ (}\inf X \textit{ is a lower bound)}$$

$$\forall y > \inf X \ \exists x \in X \textit{ s.t. } x < y \textit{ (there is no greater lower bound)}$$

If X is not bounded above, write $\sup X = \infty$. If X is not bounded below, write $\inf X = -\infty$. Convention: $\sup \emptyset = -\infty$, $\inf \emptyset = +\infty$.

The Supremum Property

The Supremum Property: Every nonempty set of real numbers that is bounded above has a supremum, which is a real number. Every nonempty set of real numbers that is bounded below has an infimum, which is a real number.

Note: $\sup X$ need not be an element of X. For example, $\sup(0,1) = 1 \notin (0,1)$.

The Supremum Property

Theorem 2 (Theorem 6.8, plus . . .)**.** *The Supremum Property and the Completeness Axiom are equivalent.*

Proof. Assume the Completeness Axiom. Let $X \subseteq \mathbf{R}$ be a nonempty set that is bounded above. Let U be the set of all upper bounds for X. Since X is bounded above, $U \neq \emptyset$. If $x \in X$ and $u \in U$, $x \leq u$ since u is an upper bound for X. So

$$x \leq u \; \forall x \in X, u \in U$$

By the Completeness Axiom,

$$\exists \alpha \in \mathbf{R} \text{ s.t. } x \leq \alpha \leq u \quad \forall x \in X, u \in U$$

α is an upper bound for X, and it is less than or equal to every other upper bound for X, so it is the least upper bound for X, so $\sup X = \alpha \in \mathbf{R}$. The case in which X is bounded below is similar. Thus, the Supremum Property holds.

Conversely, assume the Supremum Property. Suppose $L, H \subseteq \mathbf{R}$, $L \neq \emptyset \neq H$, and

$$\ell \leq h \ \forall \ell \in L, h \in H$$

Since $L \neq \emptyset$ and L is bounded above (by any element of H), $\alpha = \sup L$ exists and is real. By the definition of supremum, α is an upper bound for L, so

$$\ell \leq \alpha \ \forall \ell \in L$$

Suppose $h \in H$. Then h is an upper bound for L, so by the definition of supremum, $\alpha \leq h$. Therefore, we have shown that

$$\ell \leq \alpha \leq h \ \forall \ell \in L, h \in H$$

so the Completeness Axiom holds. □

Archimedean Property

Theorem 3 (Archimedean Property, Theorem 6.10 + ...)**.**

$$\forall x, y \in \mathbf{R}, y > 0 \ \exists n \in \mathbf{N} \ \textit{s.t.} \ ny = \underbrace{(y + \cdots + y)}_{n \ \textit{times}} > x$$

Proof. Exercise. This is a nice exercise in proof by contradiction, using the Supremum Property. □

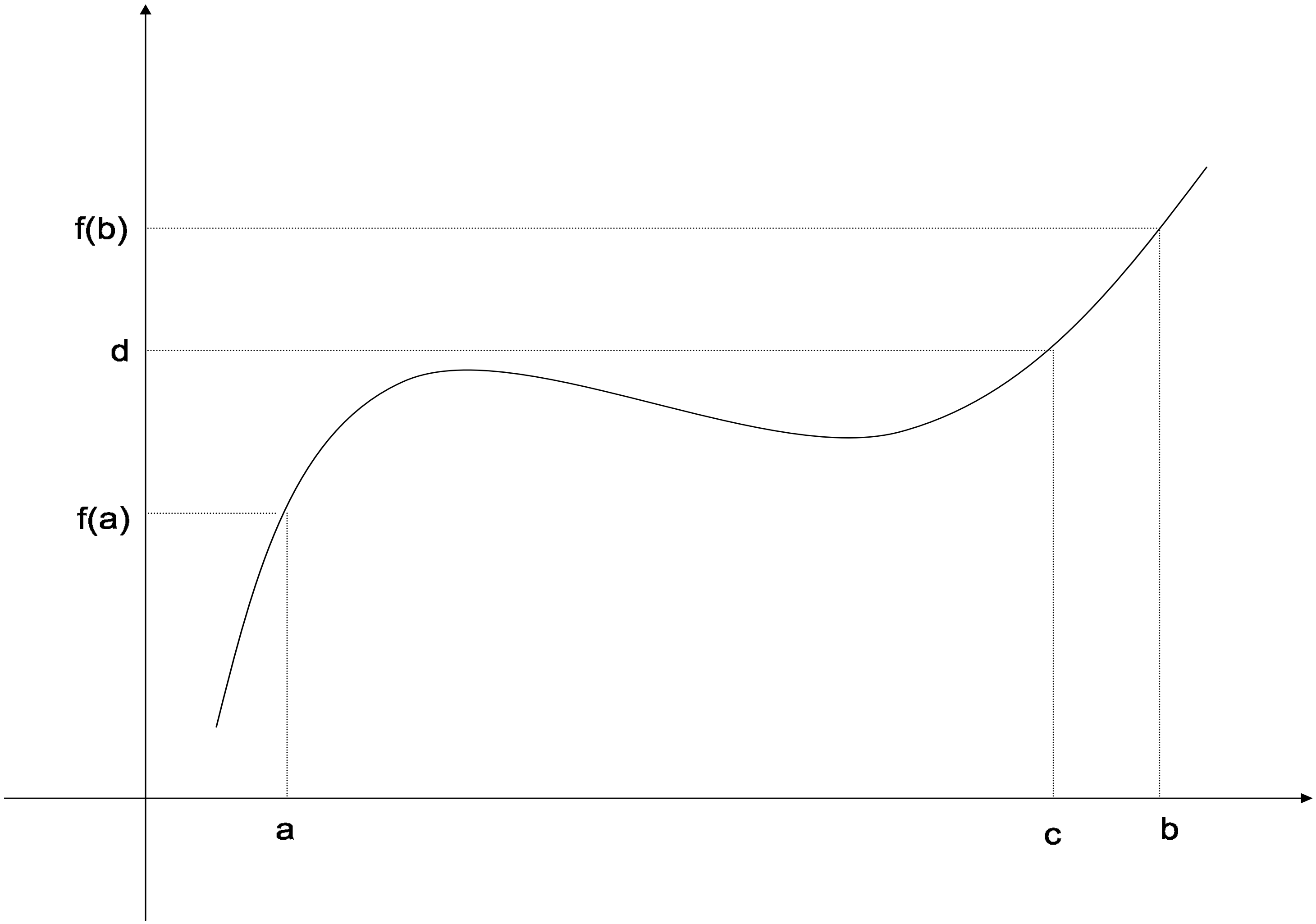
Intermediate Value Theorem

Theorem 4 (Intermediate Value Theorem)**.** *Suppose* $f : [a, b] \to \mathbf{R}$ *is continuous, and* $f(a) < d < f(b)$*. Then there exists* $c \in (a, b)$ *such that* $f(c) = d$*.*

Proof. Later, we will give a slick proof. Here, we give a bare-hands proof using the Supremum Property. Let

$$B = \{x \in [a, b] : f(x) < d\}$$

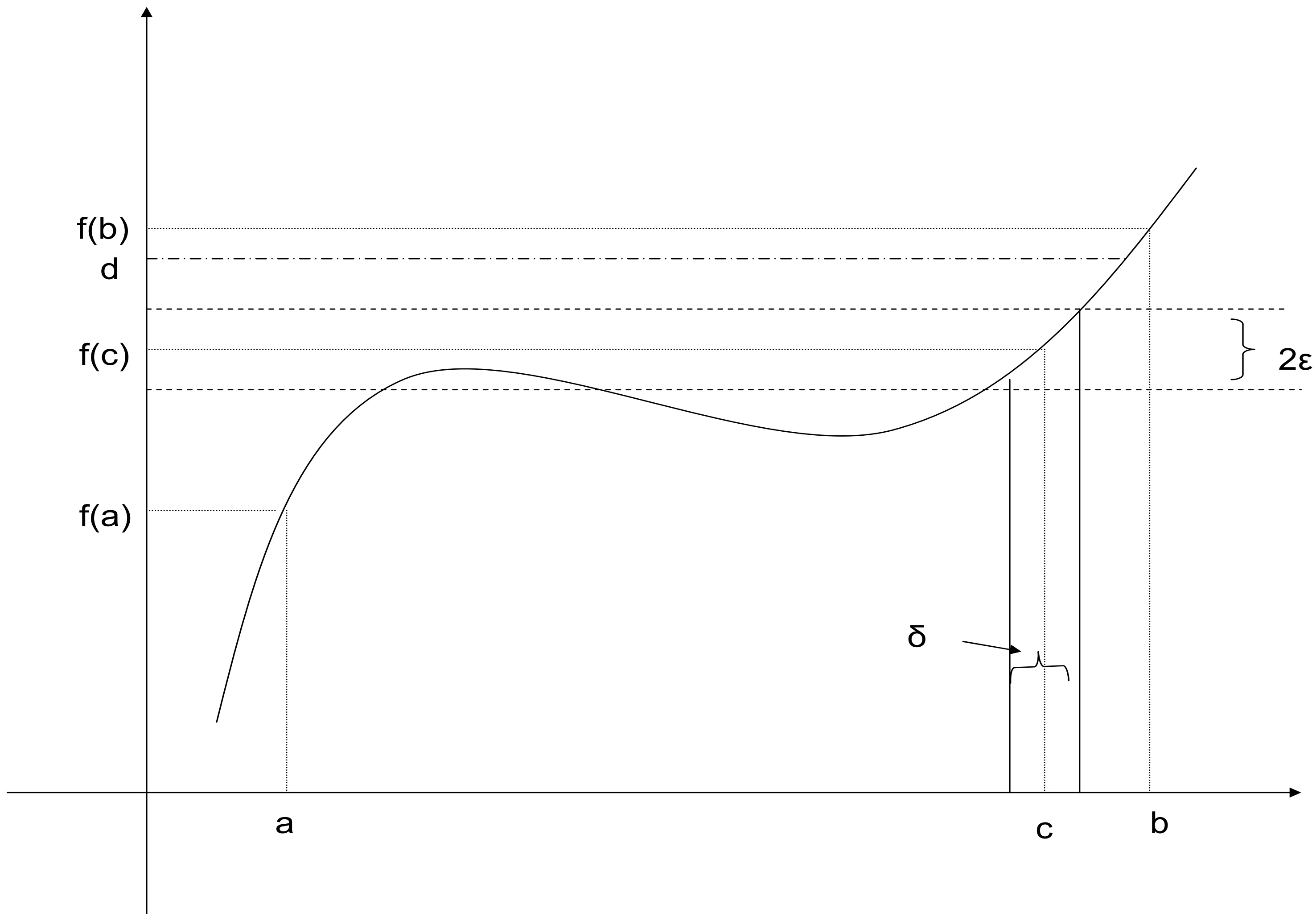
$a \in B$, so $B \neq \emptyset$; $B \subseteq [a, b]$, so B is bounded above. By the Supremum Property, $\sup B$ exists and is real so let $c = \sup B$. Since $a \in B$, $c \geq a$. $B \subseteq [a, b]$, so $c \leq b$. Therefore, $c \in [a, b]$.


f(b)
d
f(a)
a
c
b

We claim that $f(c) = d$. If not, suppose $f(c) < d$. Then since $f(b) > d$, $c \neq b$, so $c < b$. Let $\varepsilon = \frac{d-f(c)}{2} > 0$. Since f is continuous at c, there exists $\delta > 0$ such that

$$\begin{array}{rclcl} |x-c| < \delta & \Rightarrow & |f(x) - f(c)| & < & \varepsilon \\ & \Rightarrow & f(x) & < & f(c) + \varepsilon \\ & & & = & f(c) + \frac{d-f(c)}{2} \\ & & & = & \frac{f(c)+d}{2} \\ & & & < & \frac{d+d}{2} \\ & & & = & d \end{array}$$

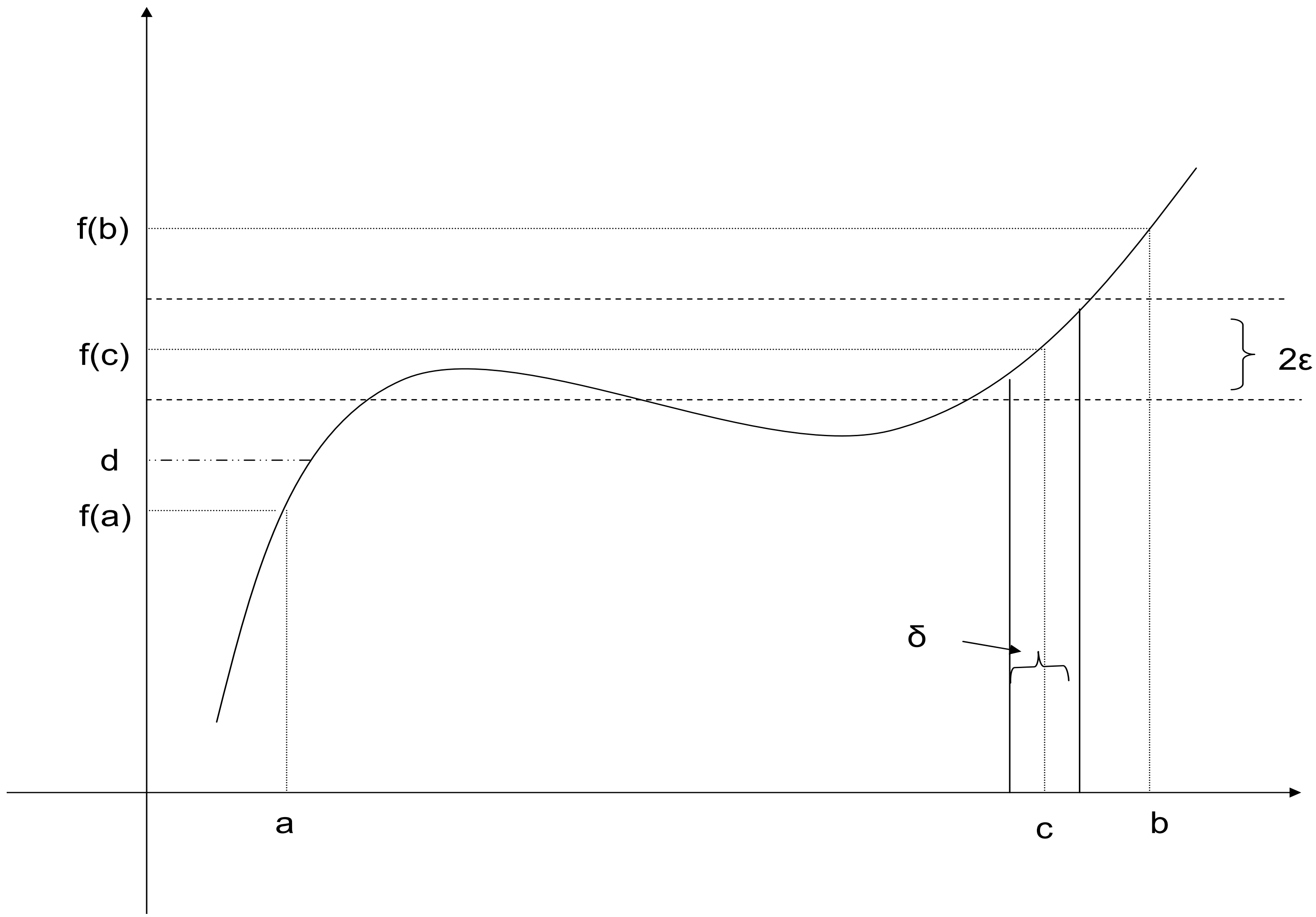
so $(c, c+\delta) \subseteq B$, so $c \neq \sup B$, contradiction.


f(b)
d
f(c)
f(a)
2ε
δ
a
c
b

Suppose $f(c) > d$. Then since $f(a) < d$, $a \neq c$, so $c > a$. Let $\varepsilon = \frac{f(c)-d}{2} > 0$. Since f is continuous at c, there exists $\delta > 0$ such that

$$\begin{array}{rcrcl} |x-c| < \delta & \Rightarrow & |f(x) - f(c)| & < & \varepsilon \\ & \Rightarrow & f(x) & > & f(c) - \varepsilon \\ & & & = & f(c) - \frac{f(c)-d}{2} \\ & & & = & \frac{f(c)+d}{2} \\ & & & > & \frac{d+d}{2} \\ & & & = & d \end{array}$$

so $(c-\delta, c+\delta) \cap B = \emptyset$. So either there exists $x \in B$ with $x \geq c+\delta$ (in which case c is not an upper bound for B) or $c-\delta$ is an upper bound for B (in which case c is not the least upper bound for B); in either case, $c \neq \sup B$, contradiction.


f(b)
f(c)
2ε
d
f(a)
δ
a
c
b

Since $f(c) \not< d$, $f(c) \not> d$, and the order is complete, $f(c) = d$. Since $f(a) < d$ and $f(b) > d$, $a \neq c \neq b$, so $c \in (a, b)$. $\square$

Corollary 1. *There exists* $x \in \mathbf{R}$ *such that* $x^2 = 2$.

Proof. Let $f(x) = x^2$, for $x \in [0, 2]$. f is continuous (Why?). $f(0) = 0 < 2$ and $f(2) = 4 > 2$, so by the Intermediate Value Theorem, there exists $c \in (0, 2)$ such that $f(c) = 2$, i.e. such that $c^2 = 2$. □