

Econ 204 2021

Lecture 5

Outline

1. Properties of Real Functions (Sect. 2.6, cont.)
2. Monotonic Functions
3. Cauchy Sequences and Complete Metric Spaces
4. Contraction Mappings
5. Contraction Mapping Theorem

Announcements

- PS 1 due today at 1pm
- solutions posted \approx 2pm today
- PS 2 due Tuesday 8/3
- Zoom information for lectures next week in bCourses

Properties of Real Functions

Here we first study properties of functions from \mathbf{R} to \mathbf{R} , making use of the additional structure we have in \mathbf{R} as opposed to general metric spaces.

Let $f : X \rightarrow \mathbf{R}$ where $X \subseteq \mathbf{R}$. We say f is *bounded above* if

$$f(X) = \{r \in \mathbf{R} : f(x) = r \text{ for some } x \in X\}$$

is bounded above. Similarly, we say f is *bounded below* if $f(X)$ is bounded below. Finally, f is *bounded* if f is both bounded above and bounded below, that is, if $f(X)$ is a bounded set.

Extreme Value Theorem

Theorem 1 (Thm. 6.23, Extreme Value Theorem). *Let $a, b \in \mathbf{R}$ with $a \leq b$ and let $f : [a, b] \rightarrow \mathbf{R}$ be a continuous function. Then f assumes its minimum and maximum on $[a, b]$. That is, if*

$$M = \sup_{t \in [a, b]} f(t) \quad m = \inf_{t \in [a, b]} f(t)$$

then $\exists t_M, t_m \in [a, b]$ such that $f(t_M) = M$ and $f(t_m) = m$.

Proof. Let

$$M = \sup\{f(t) : t \in [a, b]\}$$

If M is finite, then for each $n \in \mathbf{N}$, we may choose $t_n \in [a, b]$ such that $M \geq f(t_n) \geq M - \frac{1}{n}$ (if we couldn't make such a choice, then $M - \frac{1}{n}$ would be an upper bound and M would not be the

$$\{f(t_n)\} \subseteq \mathbb{R} \quad f(t_n) \rightarrow M$$

supremum). If M is infinite, choose $t_n \in [a, b]$ such that $f(t_n) \geq n$. By the Bolzano-Weierstrass Theorem, $\{t_n\}$ contains a convergent subsequence $\{t_{n_k}\}$, with

$$\lim_{k \rightarrow \infty} t_{n_k} = t_0 \in [a, b]$$

Since f is continuous,

$$\begin{aligned} f(t_0) &= \lim_{t \rightarrow t_0} f(t) \\ &= \lim_{k \rightarrow \infty} f(t_{n_k}) \\ &= M \end{aligned}$$

by construction,
 $f(t_n) \rightarrow M$

so M is finite and

$$f(t_0) = M = \sup\{f(t) : t \in [a, b]\}$$

so f attains its maximum and is bounded above.

The argument for the minimum is similar. □

Intermediate Value Theorem Redux

Theorem 2 (Thm. 6.24, Intermediate Value Theorem). Suppose $f : [a, b] \xrightarrow{\subseteq \mathbb{R}} \mathbf{R}$ is continuous, and $f(a) < d < f(b)$. Then there exists $c \in (a, b)$ such that $f(c) = d$.

Proof. Let

$$B = \{t \in [a, b] : f(t) < d\}$$

$a \in B$, so $B \neq \emptyset$. By the Supremum Property, $\sup B$ exists and is real so let $c = \sup B$. Since $a \in B$, $c \geq a$. $B \subseteq [a, b]$, so $c \leq b$. Therefore, $c \in [a, b]$. We claim that $f(c) = d$.

Let

$$t_n = \min \left\{ c + \frac{1}{n}, b \right\} \geq c \quad \forall n \in \mathbb{N}$$

Either $t_n > c$, in which case $t_n \notin B$, or $t_n = c$, in which case $t_n = b$ so $f(t_n) > d$, so again $t_n \notin B$; in either case, $f(t_n) \geq d$. Since f is continuous at c , $f(c) = \lim_{n \rightarrow \infty} f(t_n) \geq d$ (Theorem 3.5 in de la Fuente).

\uparrow
($t_n \rightarrow c$ by construction)

Since $c = \sup B$, we may find $s_n \in B$ such that

$$c \geq s_n \geq c - \frac{1}{n}$$

since \mathbb{N}

Since $s_n \in B$, $f(s_n) < d$. Since f is continuous at c , $f(c) = \lim_{n \rightarrow \infty} f(s_n) \leq d$ (Theorem 3.5 in de la Fuente).

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($s_n \rightarrow c$ by construction)

Since $d \leq f(c) \leq d$, $f(c) = d$. Since $f(a) < d$ and $f(b) > d$, $a \neq c \neq b$, so $c \in (a, b)$. \square

Monotonic Functions

Definition 1. A function $f : \mathbf{R} \rightarrow \mathbf{R}$ is monotonically increasing if

$$y > x \Rightarrow f(y) \geq f(x)$$

Monotonic functions are very well-behaved...

$$f(t) = \begin{cases} 1 & t \in \mathbb{Q} \\ 0 & t \notin \mathbb{Q} \end{cases}$$

Monotonic Functions

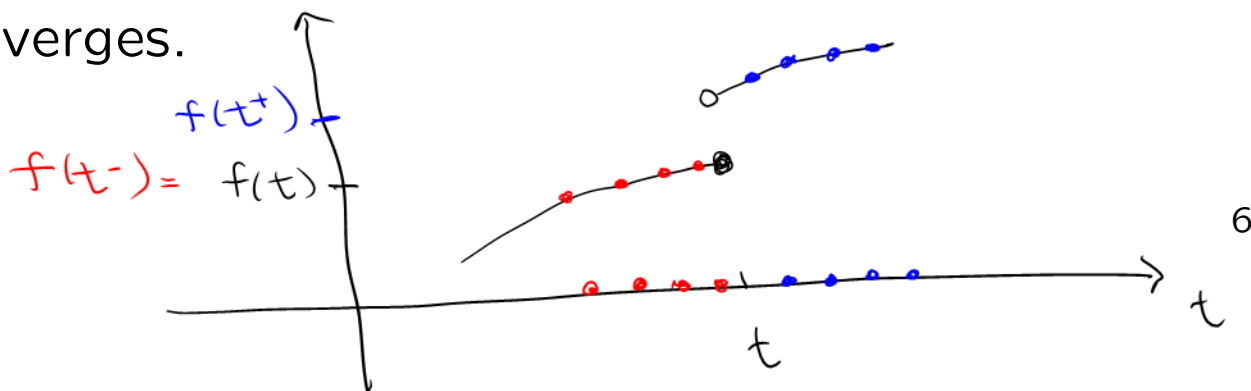
Theorem 3 (Thm. 6.27). Let $a, b \in \mathbb{R}$ with $a < b$, and let $f : (a, b) \rightarrow \mathbb{R}$ be monotonically increasing. Then the one-sided limits

right-hand limit $f(t^+) = \lim_{u \rightarrow t^+} f(u)$

left-hand limit $f(t^-) = \lim_{u \rightarrow t^-} f(u)$

exist and are real numbers for all $t \in (a, b)$.

Proof. This is analogous to the proof that a bounded monotone sequence converges. □

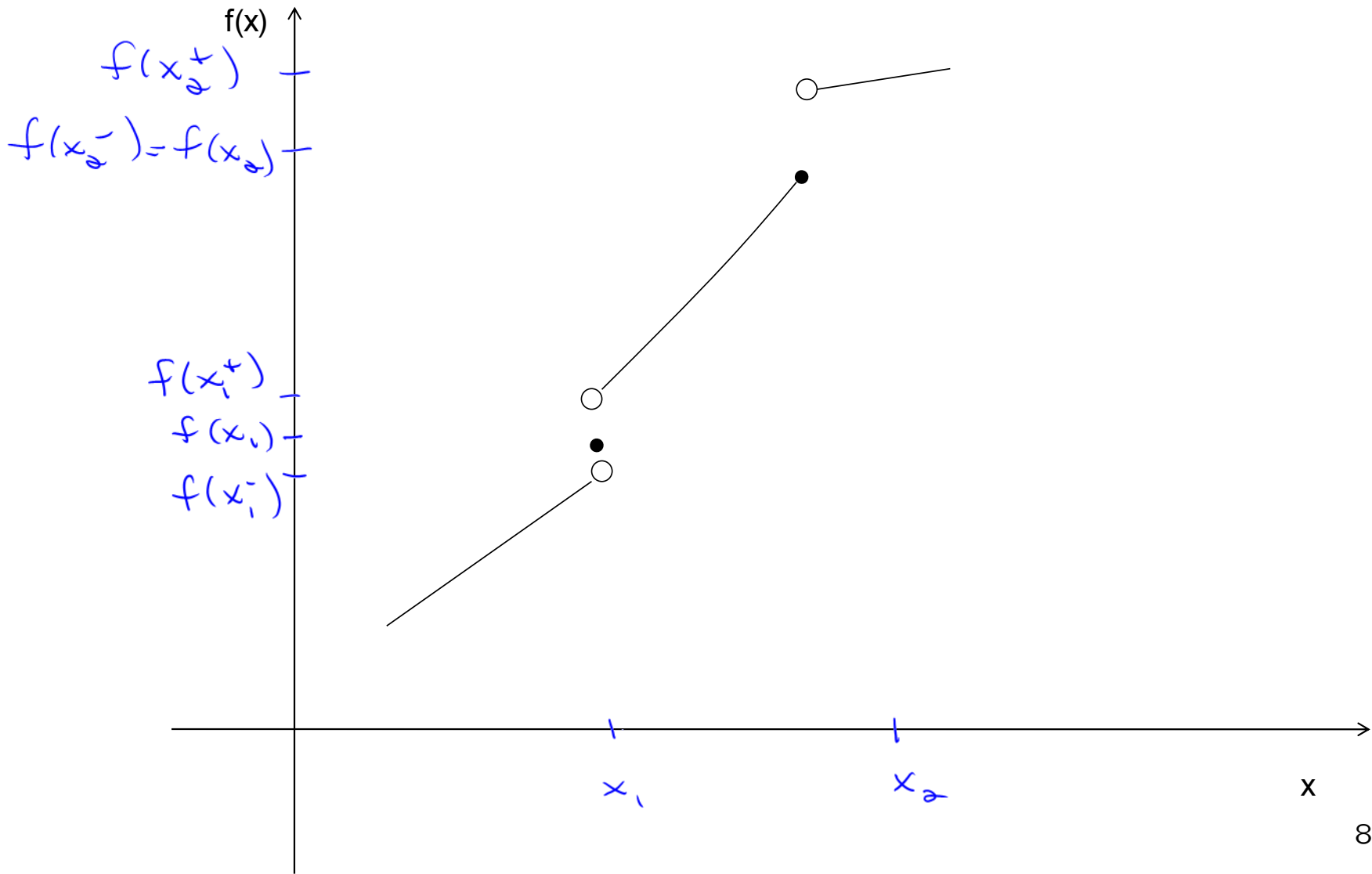


Monotonic Functions

We say that f has a *simple jump discontinuity* at t if the one-sided limits $f(t^-)$ and $f(t^+)$ both exist but f is not continuous at t .

Note that there are two ways f can have a simple jump discontinuity at t : either $f(t^+) \neq f(t^-)$, or $f(t^+) = f(t^-) \neq f(t)$.

The previous theorem says that monotonic functions have **only** simple jump discontinuities. Note that monotonicity also implies that $f(t^-) \leq f(t) \leq f(t^+)$. So a monotonic function has a discontinuity at t if and only if $f(t^+) \neq f(t^-)$.



Monotonic Functions

A monotonic function is continuous “almost everywhere” – except for at most countably many points.

Theorem 4 (Thm. 6.28). *Let $a, b \in \mathbf{R}$ with $a < b$, and let $f : (a, b) \rightarrow \mathbf{R}$ be monotonically increasing. Then*

$$D = \{t \in (a, b) : f \text{ is discontinuous at } t\}$$

is finite (possibly empty) or countable.

Proof. If $t \in D$, then $f(t^-) < f(t^+)$ (if the left- and right-hand limits agreed, then by monotonicity they would have to equal $f(t)$, so f would be continuous at t). \mathbf{Q} is dense in \mathbf{R} , that is, if

$x, y \in \mathbf{R}$ and $x < y$ then $\exists r \in \mathbf{Q}$ such that $x < r < y$. So for every $t \in D$ we may choose $r(t) \in \mathbf{Q}$ such that

$$f(t^-) < r(t) < f(t^+)$$

This defines a function $r : D \rightarrow \mathbf{Q}$. Notice that

$$s > t \Rightarrow f(s^-) \geq f(t^+)$$

so

$$s > t, s, t \in D \Rightarrow r(s) > f(s^-) \geq f(t^+) > r(t)$$

so $r(s) \neq r(t)$. Therefore, r is one-to-one, so it is a bijection from D to a subset of \mathbf{Q} . Thus D is finite or countable. \square

\uparrow
countable

Cauchy Sequences and Complete Metric Spaces

Roughly, a metric space is complete if “every sequence that ought to converge to a limit has a limit to converge to.”

Recall that $x_n \rightarrow x$ means

$$\forall \varepsilon > 0 \exists N(\varepsilon/2) \text{ s.t. } n > N(\varepsilon/2) \Rightarrow d(x_n, x) < \frac{\varepsilon}{2}$$

Observe that if $n, m > N(\varepsilon/2)$, then

$$d(x_n, x_m) \leq d(x_n, x) + d(x, x_m) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$



Cauchy Sequences and Complete Metric Spaces

This motivates the following definition:

Definition 2. *A sequence $\{x_n\}$ in a metric space (X, d) is Cauchy if*

$$\forall \varepsilon > 0 \exists N(\varepsilon) \text{ s.t. } n, m > N(\varepsilon) \Rightarrow d(x_n, x_m) < \varepsilon$$

A Cauchy sequence is trying really hard to converge, but there may not be anything for it to converge to.

Cauchy Sequences and Complete Metric Spaces

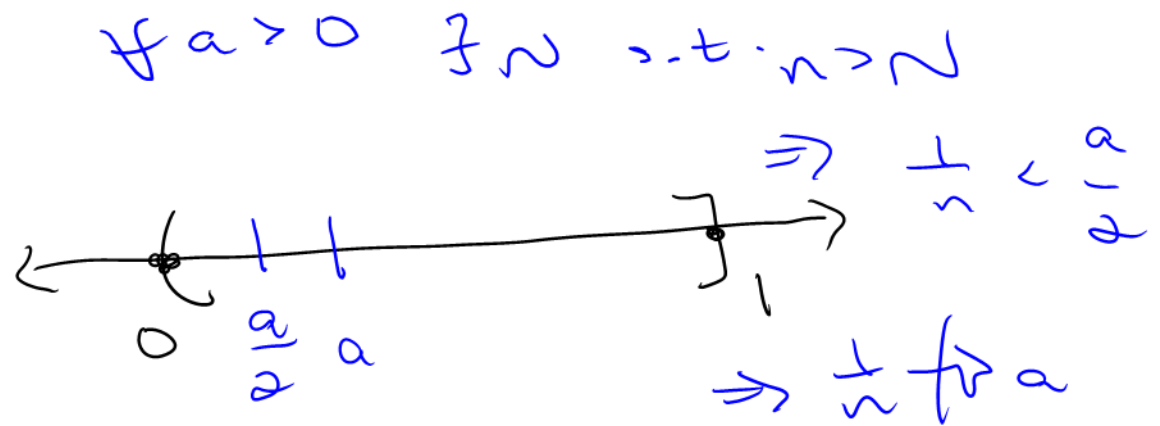
Any sequence that **does** converge must be Cauchy:

Theorem 5 (Thm. 7.2). *Every convergent sequence in a metric space is Cauchy.*

Proof. We just did it: Let $x_n \rightarrow x$. For every $\varepsilon > 0 \exists N$ such that $n > N \Rightarrow d(x_n, x) < \varepsilon/2$. Then

$$m, n > N \Rightarrow d(x_n, x_m) \leq d(x_n, x) + d(x, x_m) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

□



Example: Let $X = (0, 1]$ and d be the Euclidean metric. Let $\{x_n\} = \{\frac{1}{n}\}$. Then $x_n \rightarrow 0$ in \mathbf{E}^1 , so $\{x_n\}$ is Cauchy in \mathbf{E}^1 . Thus $\{x_n\}$ is Cauchy in (X, d) . But $\{x_n\}$ does not converge in (X, d) .

The Cauchy property depends only on the sequence and the metric d , not on the ambient metric space:

$\{x_n\}$ is Cauchy in (X, d) , but $\{x_n\}$ does not **converge** in (X, d) because the point it is trying to converge to (0) is not an element of X .

$$x_n = \left\{ \begin{array}{l} 1 \\ \dots \\ n \end{array} \right.$$

Complete Metric Spaces and Banach Spaces

Where does every Cauchy sequence get what it wants?

Definition 3. A metric space (X, d) is complete if every Cauchy sequence $\{x_n\} \subseteq X$ converges to a limit $x \in X$.

Definition 4. A Banach space is a normed vector space that is complete in the metric generated by its norm.

Complete Metric Spaces and Banach Spaces

Example: Consider the earlier example of $X = (0, 1]$ with d the usual Euclidean metric. The sequence $\{x_n\}$ with $x_n = \frac{1}{n}$ is Cauchy but does not converge, so $((0, 1], d)$ is not complete.

Example: \mathbb{Q} is not complete in the Euclidean metric. To see this, let

$$x_n = \frac{\lfloor 10^n \sqrt{2} \rfloor}{10^n}$$

where $\lfloor y \rfloor$ is the greatest integer less than or equal to y ; x_n is just equal to the decimal expansion of $\sqrt{2}$ to n digits past the decimal point. Clearly, x_n is rational. $|x_n - \sqrt{2}| \leq 10^{-n}$, so $x_n \rightarrow \sqrt{2}$ in \mathbb{E}^1 , so $\{x_n\}$ is Cauchy in \mathbb{E}^1 , hence Cauchy in \mathbb{Q} ; since $\sqrt{2} \notin \mathbb{Q}$, $\{x_n\}$ is not convergent in \mathbb{Q} , so \mathbb{Q} is not complete.

Complete Metric Spaces and Banach Spaces

Theorem 6 (Thm. 7.10). \mathbf{R} is complete with the usual metric (so \mathbf{E}^1 is a Banach space).

Proof. Suppose $\{x_n\}$ is a Cauchy sequence in \mathbf{R} . Fix $\varepsilon > 0$. Find $N(\varepsilon/2)$ such that

$$n, m > N(\varepsilon/2) \Rightarrow |x_n - x_m| < \frac{\varepsilon}{2}$$

Let

$$\begin{aligned}\alpha_n &= \sup\{x_k : k \geq n\} \\ \beta_n &= \inf\{x_k : k \geq n\}\end{aligned}$$

Fix $m > N(\varepsilon/2)$. Then

$$\begin{aligned}k \geq m &\Rightarrow k > N(\varepsilon/2) \Rightarrow x_k < x_m + \frac{\varepsilon}{2} \\ &\Rightarrow \alpha_m = \sup\{x_k : k \geq m\} \leq x_m + \frac{\varepsilon}{2}\end{aligned}$$

Since $\alpha_m < \infty$,

$$\limsup x_n = \lim_{n \rightarrow \infty} \alpha_n \leq \alpha_m \leq x_m + \frac{\varepsilon}{2}$$

since the sequence $\{\alpha_n\}$ is decreasing. Similarly,

$$\liminf x_n \geq x_m - \frac{\varepsilon}{2}$$

Therefore, $x_m - \frac{\varepsilon}{2} \leq \liminf x_n \leq \limsup x_n \leq x_m + \frac{\varepsilon}{2}$

$$\Rightarrow 0 \leq \limsup_{n \rightarrow \infty} x_n - \liminf_{n \rightarrow \infty} x_n \leq \varepsilon$$

Since ε is arbitrary,

$$\limsup_{n \rightarrow \infty} x_n = \liminf_{n \rightarrow \infty} x_n \in \mathbf{R}$$

Thus $\lim_{n \rightarrow \infty} x_n$ exists and is real, so $\{x_n\}$ is convergent. \square

Complete Metric Spaces and Banach Spaces

Theorem 7 (Thm. 7.11). \mathbf{E}^n is complete for every $n \in \mathbf{N}$.

Proof. See de la Fuente.



Complete Metric Spaces and Banach Spaces

Theorem 8 (Thm. 7.9). *Suppose (X, d) is a complete metric space and $Y \subseteq X$. Then $(Y, d) = (Y, d|_Y)$ is complete if and only if Y is a closed subset of X .*

\Rightarrow *Proof.* Suppose (Y, d) is complete. We need to show that Y is closed. Consider a sequence $\{y_n\} \subseteq Y$ such that $y_n \rightarrow_{(X, d)} x \in X$. Then $\{y_n\}$ is Cauchy in X , hence Cauchy in Y ; since Y is complete, $y_n \rightarrow_{(Y, d)} y$ for some $y \in Y$. Therefore, $y_n \rightarrow_{(X, d)} y$. By uniqueness of limits, $y = x$, so $x \in Y$. Thus Y is closed.

\Leftarrow Conversely, suppose Y is closed. We need to show that Y is complete. Let $\{y_n\}$ be a Cauchy sequence in Y . Then $\{y_n\}$ is Cauchy in X , hence convergent, so $y_n \rightarrow_{(X, d)} x$ for some $x \in X$. Since Y is closed, $x \in Y$, so $y_n \rightarrow_{(Y, d)} x \in Y$. Thus Y is complete. \square

Complete Metric Spaces and Banach Spaces

Theorem 9 (Thm. 7.12). *Given $X \subseteq \mathbf{R}^n$, let $C(X)$ be the set of bounded continuous functions from X to \mathbf{R} with*

$$\|f\|_{\infty} = \sup\{|f(x)| : x \in X\}$$

Then $C(X)$ is a Banach space.

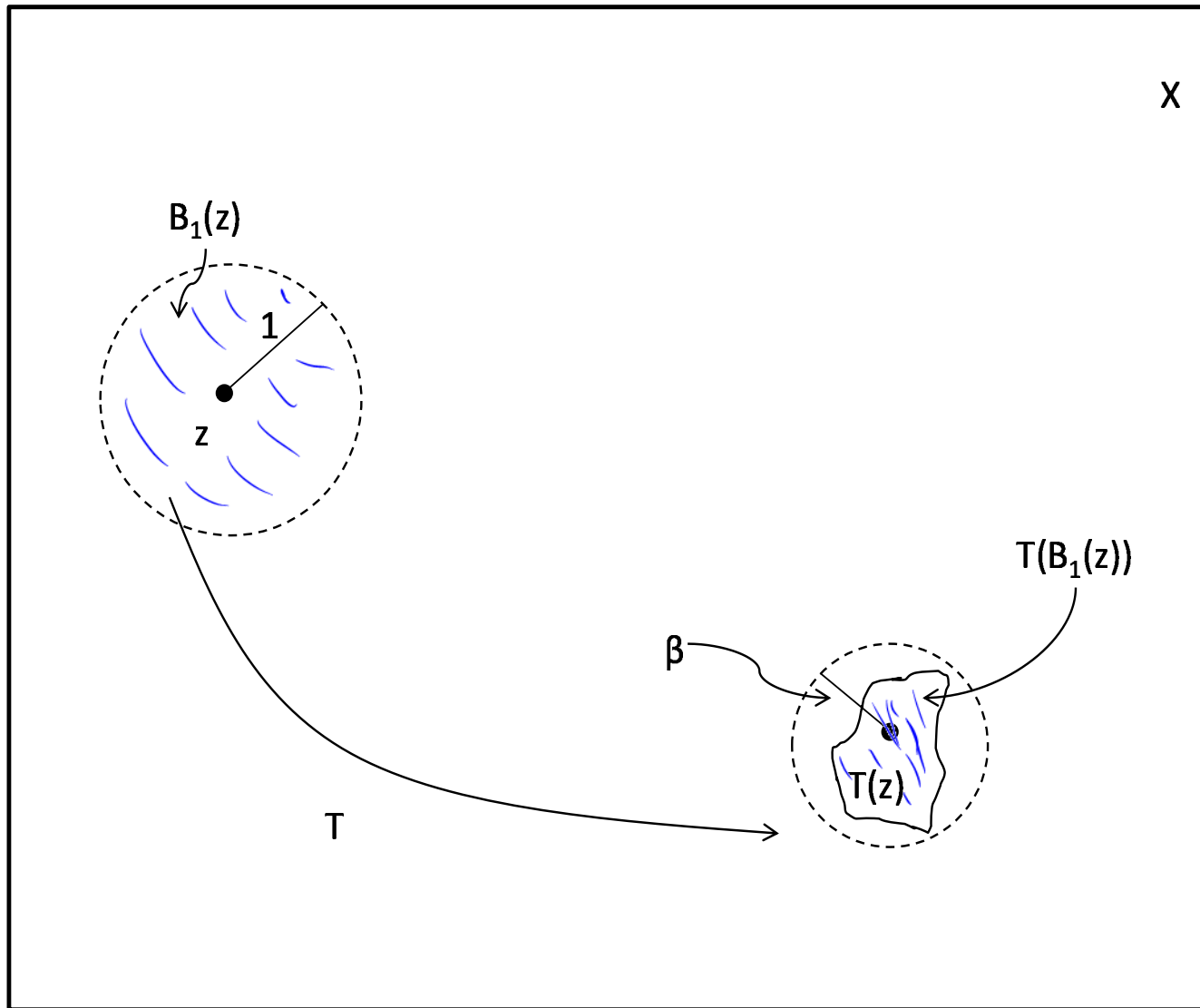
Contractions

Definition 5. Let (X, d) be a nonempty complete metric space. An operator is a function $T : X \rightarrow X$.

An operator T is a contraction of modulus β if $0 \leq \beta < 1$ and

$$d(T(x), T(y)) \leq \beta d(x, y) \quad \forall x, y \in X$$

A contraction shrinks distances by a **uniform** factor $\beta < 1$.



Contractions

Theorem 10. *Every contraction is uniformly continuous.*

Proof. Fix $\varepsilon > 0$. Let $\delta = \frac{\varepsilon}{\beta}$. Then $\forall x, y$ such that $d(x, y) < \delta$,

$$d(T(x), T(y)) \leq \beta d(x, y) < \beta \delta = \varepsilon$$

□

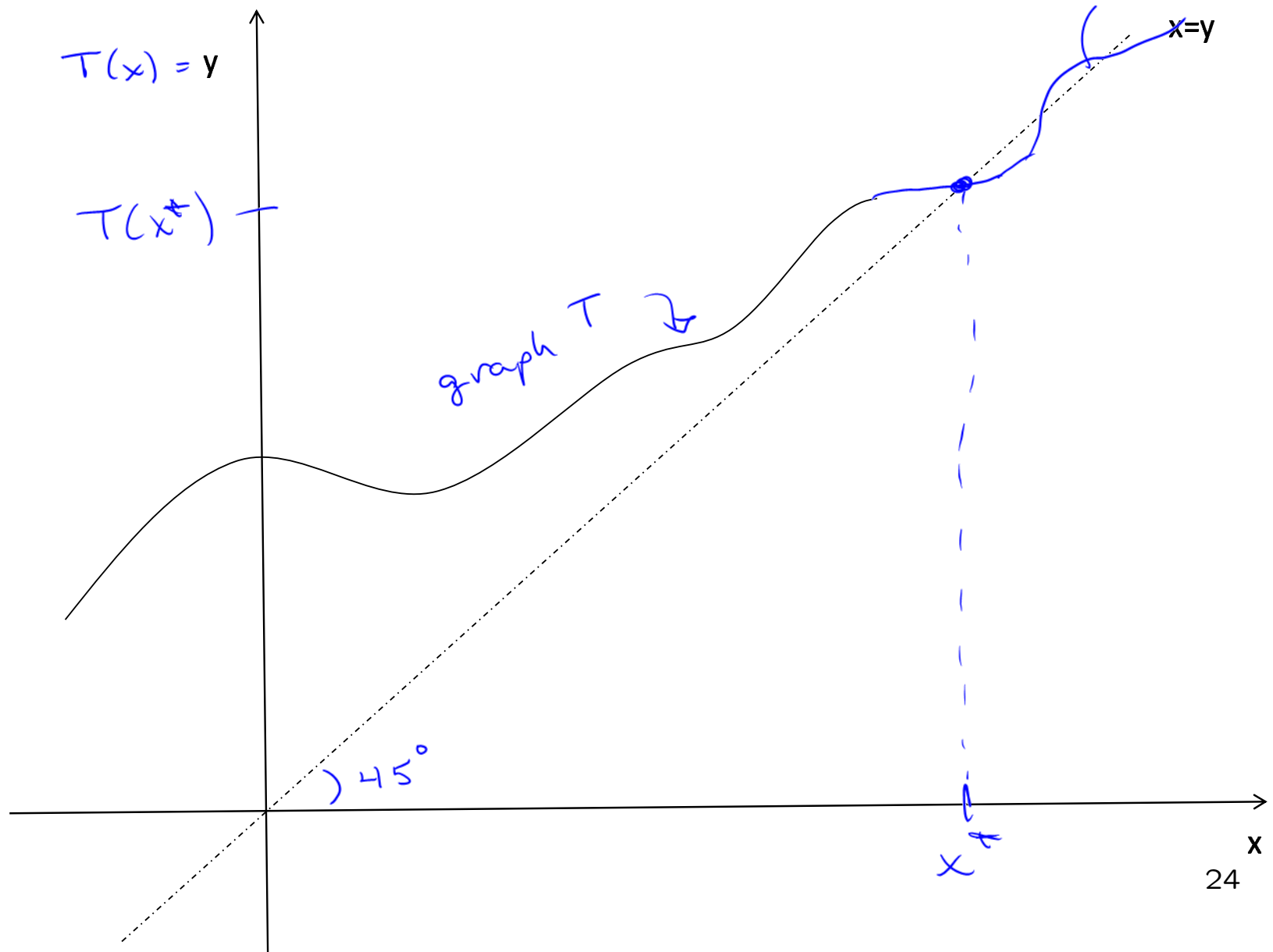
Note that a contraction is Lipschitz continuous with Lipschitz constant $\beta < 1$ (and hence also uniformly continuous).

Contractions and Fixed Points

Definition 6. A fixed point of an operator T is point $x^* \in X$ such that $T(x^*) = x^*$.

$$? \exists x^* \text{ s.t. } T(x^*) = x^*$$

$$\{(x, y) \in X \times X : x = y\}$$



Contraction Mapping Theorem

Theorem 11 (Thm. 7.16, Contraction Mapping Theorem). *Let (X, d) be a nonempty complete metric space and $T : X \rightarrow X$ a contraction with modulus $\beta < 1$. Then*

1. *T has a unique fixed point x^* .*

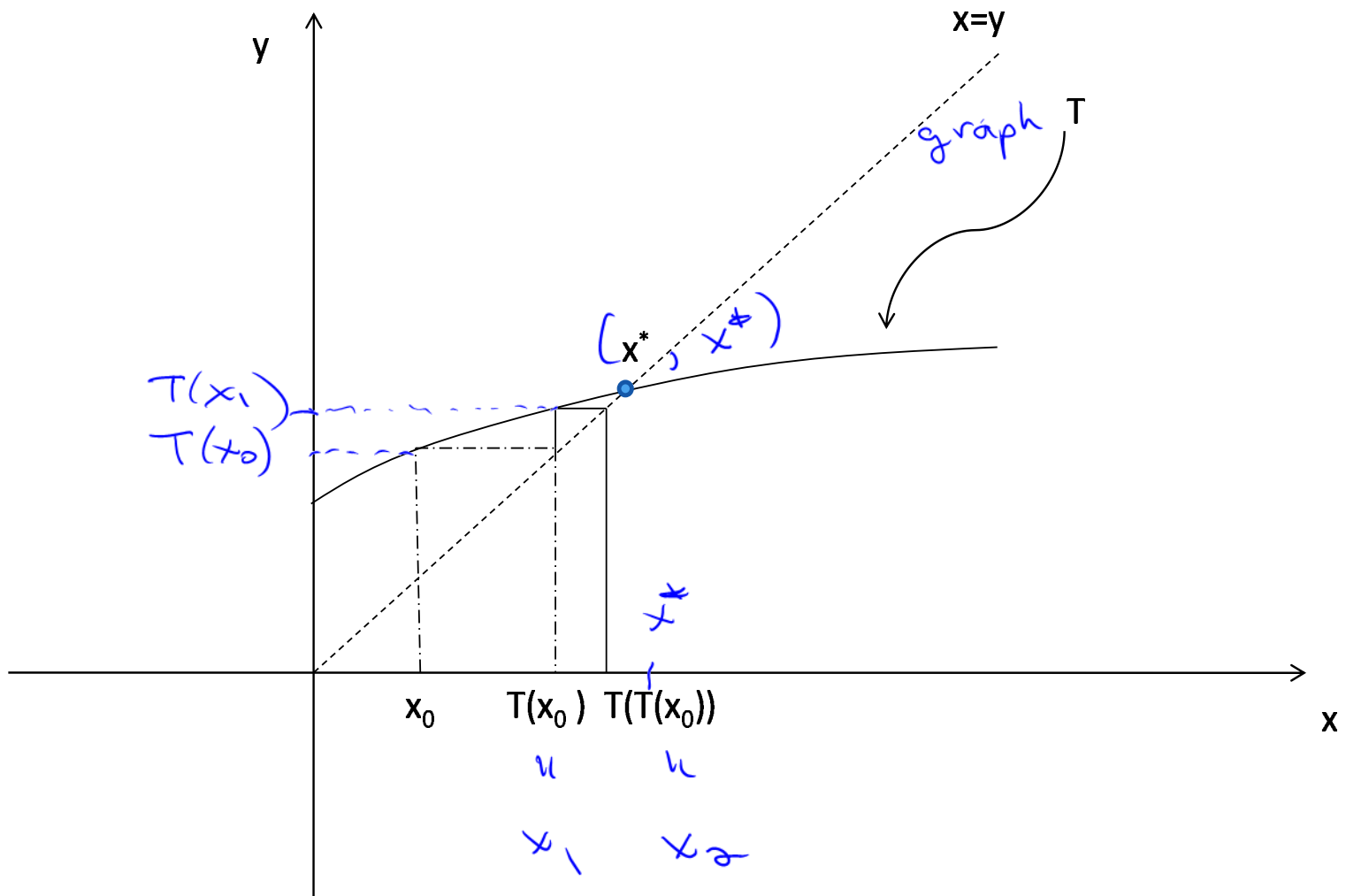
2. *For every $x_0 \in X$, the sequence $\{x_n\}$ where*

$x_1 = T(x_0), x_2 = T(x_1) = T(T(x_0)), \dots, x_n = T(x_{n-1})$ for each n converges to x^ .*

Note that the theorem asserts both the existence and uniqueness of the fixed point, as well as giving an algorithm to find the fixed point of a contraction.

Also note that the algorithm generates a sequence that converges to the fixed point for any initial point x_0 .

Later in the course we will discuss more general fixed point theorems which, in contrast, only guarantee existence, and are not constructive.



Proof. Define the sequence $\{x_n\}$ as above by first fixing $x_0 \in X$ and then letting $x_n = T(x_{n-1}) = T^n(x_0)$ for $n = 1, 2, \dots$, where $T^n = T \circ T \circ \dots \circ T$ is the n -fold iteration of T . We first show that $\{x_n\}$ is Cauchy, and hence converges to a limit x . Then

$$\begin{aligned} d(x_{n+1}, x_n) &= d(T(x_n), T(x_{n-1})) \\ &\leq \beta d(x_n, x_{n-1}) = \beta d(T(x_{n-1}), T(x_{n-2})) \\ &\leq \beta^2 d(x_{n-1}, x_{n-2}) \\ &\vdots \\ &\leq \beta^n d(x_1, x_0) \end{aligned}$$

Then for any $n > m$,

$$\begin{aligned}d(x_n, x_m) &\leq d(x_n, x_{n-1}) + d(x_{n-1}, x_{n-2}) + \cdots + d(x_{m+1}, x_m) \\&\leq (\beta^{n-1} + \beta^{n-2} + \cdots + \beta^m)d(x_1, x_0) \\&= d(x_1, x_0) \sum_{\ell=m}^{n-1} \beta^\ell \\&< d(x_1, x_0) \sum_{\ell=m}^{\infty} \beta^\ell \\&= \frac{\beta^m}{1-\beta}d(x_1, x_0) \quad (\text{sum of a geometric series})\end{aligned}$$

Fix $\varepsilon > 0$. Since $\frac{\beta^m}{1-\beta}d(x_1, x_0) \rightarrow 0$ as $m \rightarrow \infty$, choose $N(\varepsilon)$ such that for any $m > N(\varepsilon)$, $\frac{\beta^m}{1-\beta}d(x_1, x_0) < \varepsilon$. Then for $n, m > N(\varepsilon)$,

$$d(x_n, x_m) \leq \frac{\beta^m}{1-\beta}d(x_1, x_0) < \varepsilon$$

Therefore, $\{x_n\}$ is Cauchy. Since (X, d) is complete, $x_n \rightarrow x^*$ for some $x^* \in X$.

Next, we show that x^* is a fixed point of T .

$$\begin{aligned} T(x^*) &= T\left(\lim_{n \rightarrow \infty} x_n\right) \\ &= \lim_{n \rightarrow \infty} T(x_n) \text{ since } T \text{ is continuous} \\ &= \lim_{n \rightarrow \infty} x_{n+1} \\ &= x^* \end{aligned}$$

so x^* is a fixed point of T .

Finally, we show that there is at most one fixed point. Suppose x^* and y^* are both fixed points of T , so $T(x^*) = x^*$ and $T(y^*) = y^*$.

Then

$$\begin{aligned}d(x^*, y^*) &= d(T(x^*), T(y^*)) \\ &\leq \beta d(x^*, y^*) \\ \Rightarrow (1 - \beta)d(x^*, y^*) &\leq 0 \\ \Rightarrow d(x^*, y^*) &\leq 0\end{aligned}$$

So $d(x^*, y^*) = 0$, which implies $x^* = y^*$.



Continuous Dependence on Parameters

Theorem 12. (Thm. 7.18', Continuous Dependence on Parameters) *Let (X, d) and (Ω, ρ) be two metric spaces and $T : X \times \Omega \rightarrow X$. For each $\omega \in \Omega$ let $T_\omega : X \rightarrow X$ be defined by*

$$T_\omega(x) = T(x, \omega)$$

Suppose (X, d) is complete, T is continuous in ω , that is $T(x, \cdot) : \Omega \rightarrow X$ is continuous for each $x \in X$, and $\exists \beta < 1$ such that T_ω is a contraction of modulus $\beta \quad \forall \omega \in \Omega$. Then the fixed point function $x^ : \Omega \rightarrow X$ defined by*

$$x^*(\omega) = T_\omega(x^*(\omega))$$

is continuous.

Blackwell's Sufficient Conditions

An important result due to Blackwell gives a set of sufficient conditions for an operator to be a contraction that is particularly useful in dynamic programming problems.

Let X be a set, and let $B(X)$ be the set of all bounded functions from X to \mathbf{R} . Then $(B(X), \|\cdot\|_\infty)$ is a normed vector space.

Notice that below we use shorthand notation that identifies a constant function with its constant value in \mathbf{R} , that is, we write interchangeably $a \in \mathbf{R}$ and $a : X \rightarrow \mathbf{R}$ to denote the function such that $a(x) = a \forall x \in X$.

Blackwell's Sufficient Conditions

Theorem 13. (Blackwell's Sufficient Conditions) Consider $B(X)$ with the sup norm $\| \cdot \|_\infty$. Let $T : B(X) \rightarrow B(X)$ be an operator satisfying

1. (monotonicity) $f(x) \leq g(x) \forall x \in X \Rightarrow (Tf)(x) \leq (Tg)(x) \forall x \in X$

2. (discounting) $\exists \beta \in (0, 1)$ such that for every $a \geq 0$ and $x \in X$,

$$(T(f + a))(x) \leq (Tf)(x) + \beta a$$

Then T is a contraction with modulus β .

$$u = z = \{0, 1\}$$

$$P(u, z) = \begin{cases} 0 & u \neq z \\ 1 & u = z \end{cases}$$

$\forall z \in Z$:

$$\inf_{u \in \mathcal{U}} P(u, z) = 0$$

$$V_- = \sup_{z \in Z} \inf_{u \in \mathcal{U}} P(u, z) = 0$$

$\forall u \in \mathcal{U}$: $\beta(u) = u \Rightarrow P(u, \beta(u)) = 1$

$$\|T(f) - T(g)\|_\infty \leq \beta \|f - g\|_\infty$$

Proof. Fix $f, g \in B(X)$. By the definition of the sup norm,

$$f(x) \leq g(x) + \|f - g\|_\infty \quad \forall x \in X$$

Then

$$\begin{aligned} (Tf)(x) &\leq (T(g + \|f - g\|_\infty))(x) \quad \forall x \in X && \text{(monotonicity)} \\ &\leq (Tg)(x) + \beta \|f - g\|_\infty \quad \forall x \in X && \text{(discounting)} \end{aligned}$$

Thus

$$(Tf)(x) - (Tg)(x) \leq \beta \|f - g\|_\infty \quad \forall x \in X$$

Reversing the roles of f and g above gives

$$(Tg)(x) - (Tf)(x) \leq \beta \|f - g\|_\infty \quad \forall x \in X$$

Thus

$$\|T(f) - T(g)\|_\infty \leq \beta \|f - g\|_\infty$$

Thus T is a contraction with modulus β

□