

# Econ 204 2021

## Lecture 5

### Outline

1. Properties of Real Functions (Sect. 2.6, cont.)
2. Monotonic Functions
3. Cauchy Sequences and Complete Metric Spaces
4. Contraction Mappings
5. Contraction Mapping Theorem

# Properties of Real Functions

Here we first study properties of functions from  $\mathbf{R}$  to  $\mathbf{R}$ , making use of the additional structure we have in  $\mathbf{R}$  as opposed to general metric spaces.

Let  $f : X \rightarrow \mathbf{R}$  where  $X \subseteq \mathbf{R}$ . We say  $f$  is *bounded above* if

$$f(X) = \{r \in \mathbf{R} : f(x) = r \text{ for some } x \in X\}$$

is bounded above. Similarly, we say  $f$  is *bounded below* if  $f(X)$  is bounded below. Finally,  $f$  is *bounded* if  $f$  is both bounded above and bounded below, that is, if  $f(X)$  is a bounded set.

# Extreme Value Theorem

**Theorem 1** (Thm. 6.23, Extreme Value Theorem). *Let  $a, b \in \mathbf{R}$  with  $a \leq b$  and let  $f : [a, b] \rightarrow \mathbf{R}$  be a continuous function. Then  $f$  assumes its minimum and maximum on  $[a, b]$ . That is, if*

$$M = \sup_{t \in [a, b]} f(t) \quad m = \inf_{t \in [a, b]} f(t)$$

*then  $\exists t_M, t_m \in [a, b]$  such that  $f(t_M) = M$  and  $f(t_m) = m$ .*

*Proof.* Let

$$M = \sup\{f(t) : t \in [a, b]\}$$

If  $M$  is finite, then for each  $n$ , we may choose  $t_n \in [a, b]$  such that  $M \geq f(t_n) \geq M - \frac{1}{n}$  (if we couldn't make such a choice, then  $M - \frac{1}{n}$  would be an upper bound and  $M$  would not be the

supremum). If  $M$  is infinite, choose  $t_n$  such that  $f(t_n) \geq n$ . By the Bolzano-Weierstrass Theorem,  $\{t_n\}$  contains a convergent subsequence  $\{t_{n_k}\}$ , with

$$\lim_{k \rightarrow \infty} t_{n_k} = t_0 \in [a, b]$$

Since  $f$  is continuous,

$$\begin{aligned} f(t_0) &= \lim_{t \rightarrow t_0} f(t) \\ &= \lim_{k \rightarrow \infty} f(t_{n_k}) \\ &= M \end{aligned}$$

so  $M$  is finite and

$$f(t_0) = M = \sup\{f(t) : t \in [a, b]\}$$

so  $f$  attains its maximum and is bounded above.

The argument for the minimum is similar.



## Intermediate Value Theorem Redux

**Theorem 2** (Thm. 6.24, Intermediate Value Theorem). *Suppose  $f : [a, b] \rightarrow \mathbf{R}$  is continuous, and  $f(a) < d < f(b)$ . Then there exists  $c \in (a, b)$  such that  $f(c) = d$ .*

*Proof.* Let

$$B = \{t \in [a, b] : f(t) < d\}$$

$a \in B$ , so  $B \neq \emptyset$ . By the Supremum Property,  $\sup B$  exists and is real so let  $c = \sup B$ . Since  $a \in B$ ,  $c \geq a$ .  $B \subseteq [a, b]$ , so  $c \leq b$ . Therefore,  $c \in [a, b]$ . We claim that  $f(c) = d$ .

Let

$$t_n = \min \left\{ c + \frac{1}{n}, b \right\} \geq c$$

Either  $t_n > c$ , in which case  $t_n \notin B$ , or  $t_n = c$ , in which case  $t_n = b$  so  $f(t_n) > d$ , so again  $t_n \notin B$ ; in either case,  $f(t_n) \geq d$ . Since  $f$  is continuous at  $c$ ,  $f(c) = \lim_{n \rightarrow \infty} f(t_n) \geq d$  (Theorem 3.5 in de la Fuente).

Since  $c = \sup B$ , we may find  $s_n \in B$  such that

$$c \geq s_n \geq c - \frac{1}{n}$$

Since  $s_n \in B$ ,  $f(s_n) < d$ . Since  $f$  is continuous at  $c$ ,  $f(c) = \lim_{n \rightarrow \infty} f(s_n) \leq d$  (Theorem 3.5 in de la Fuente).

Since  $d \leq f(c) \leq d$ ,  $f(c) = d$ . Since  $f(a) < d$  and  $f(b) > d$ ,  $a \neq c \neq b$ , so  $c \in (a, b)$ .  $\square$

# Monotonic Functions

**Definition 1.** A function  $f : \mathbf{R} \rightarrow \mathbf{R}$  is monotonically increasing if

$$y > x \Rightarrow f(y) \geq f(x)$$

Monotonic functions are very well-behaved...

## Monotonic Functions

**Theorem 3** (Thm. 6.27). *Let  $a, b \in \mathbb{R}$  with  $a < b$ , and let  $f : (a, b) \rightarrow \mathbb{R}$  be monotonically increasing. Then the one-sided limits*

$$f(t^+) = \lim_{u \rightarrow t^+} f(u)$$
$$f(t^-) = \lim_{u \rightarrow t^-} f(u)$$

*exist and are real numbers for all  $t \in (a, b)$ .*

*Proof.* This is analogous to the proof that a bounded monotone sequence converges. □

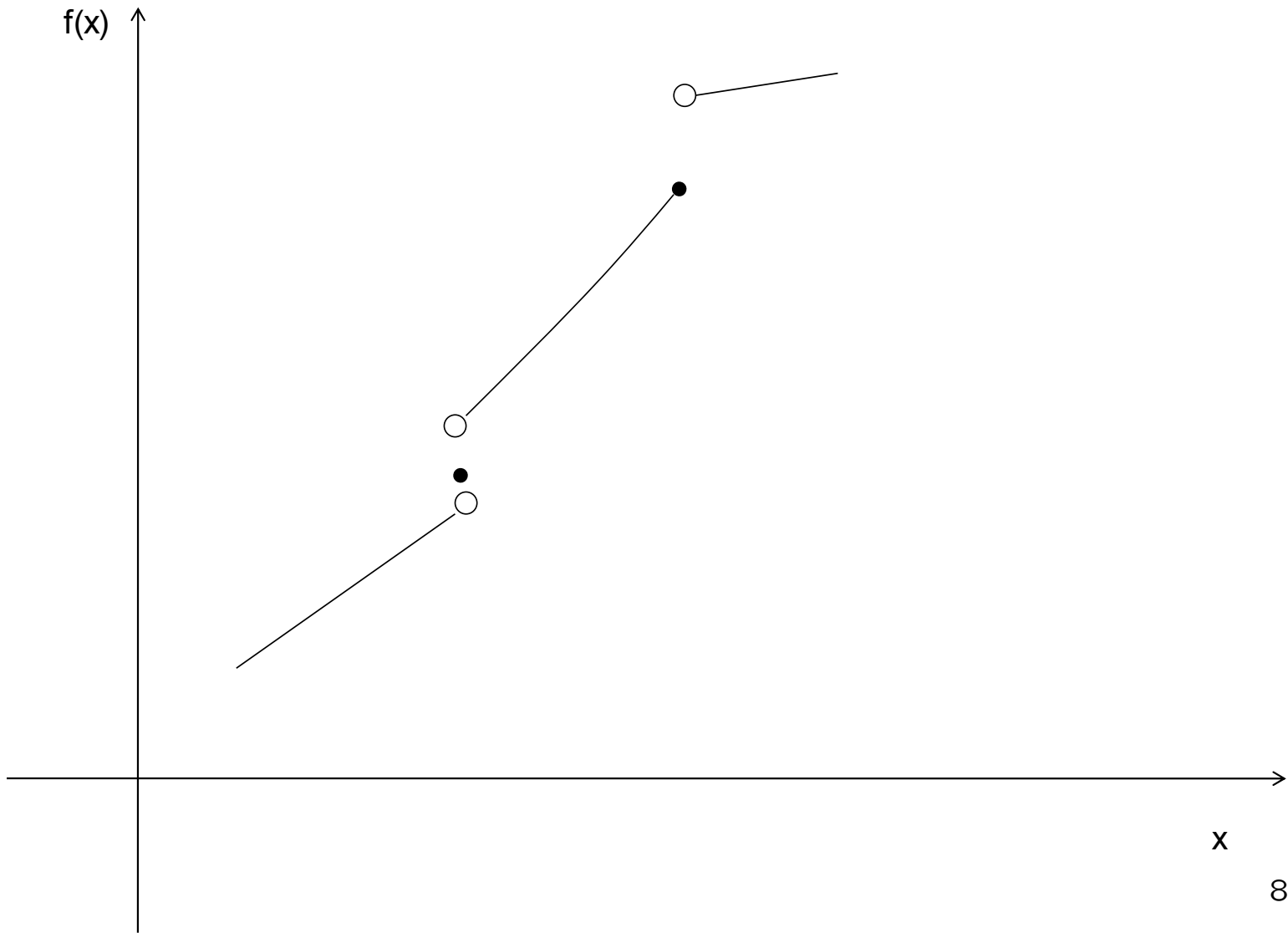


# Monotonic Functions

We say that  $f$  has a *simple jump discontinuity* at  $t$  if the one-sided limits  $f(t^-)$  and  $f(t^+)$  both exist but  $f$  is not continuous at  $t$ .

Note that there are two ways  $f$  can have a simple jump discontinuity at  $t$ : either  $f(t^+) \neq f(t^-)$ , or  $f(t^+) = f(t^-) \neq f(t)$ .

The previous theorem says that monotonic functions have **only** simple jump discontinuities. Note that monotonicity also implies that  $f(t^-) \leq f(t) \leq f(t^+)$ . So a monotonic function has a discontinuity at  $t$  if and only if  $f(t^+) \neq f(t^-)$ .



# Monotonic Functions

A monotonic function is continuous “almost everywhere” – except for at most countably many points.

**Theorem 4** (Thm. 6.28). *Let  $a, b \in \mathbf{R}$  with  $a < b$ , and let  $f : (a, b) \rightarrow \mathbf{R}$  be monotonically increasing. Then*

$$D = \{t \in (a, b) : f \text{ is discontinuous at } t\}$$

*is finite (possibly empty) or countable.*

*Proof.* If  $t \in D$ , then  $f(t^-) < f(t^+)$  (if the left- and right-hand limits agreed, then by monotonicity they would have to equal  $f(t)$ , so  $f$  would be continuous at  $t$ ).  $\mathbf{Q}$  is dense in  $\mathbf{R}$ , that is, if

$x, y \in \mathbf{R}$  and  $x < y$  then  $\exists r \in \mathbf{Q}$  such that  $x < r < y$ . So for every  $t \in D$  we may choose  $r(t) \in \mathbf{Q}$  such that

$$f(t^-) < r(t) < f(t^+)$$

This defines a function  $r : D \rightarrow \mathbf{Q}$ . Notice that

$$s > t \Rightarrow f(s^-) \geq f(t^+)$$

so

$$s > t, s, t \in D \Rightarrow r(s) > f(s^-) \geq f(t^+) > r(t)$$

so  $r(s) \neq r(t)$ . Therefore,  $r$  is one-to-one, so it is a bijection from  $D$  to a subset of  $\mathbf{Q}$ . Thus  $D$  is finite or countable.  $\square$

# Cauchy Sequences and Complete Metric Spaces

Roughly, a metric space is complete if “every sequence that ought to converge to a limit has a limit to converge to.”

Recall that  $x_n \rightarrow x$  means

$$\forall \varepsilon > 0 \exists N(\varepsilon/2) \text{ s.t. } n > N(\varepsilon/2) \Rightarrow d(x_n, x) < \frac{\varepsilon}{2}$$

Observe that if  $n, m > N(\varepsilon/2)$ , then

$$d(x_n, x_m) \leq d(x_n, x) + d(x, x_m) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

# Cauchy Sequences and Complete Metric Spaces

This motivates the following definition:

**Definition 2.** *A sequence  $\{x_n\}$  in a metric space  $(X, d)$  is Cauchy if*

$$\forall \varepsilon > 0 \exists N(\varepsilon) \text{ s.t. } n, m > N(\varepsilon) \Rightarrow d(x_n, x_m) < \varepsilon$$

A Cauchy sequence is trying really hard to converge, but there may not be anything for it to converge to.

# Cauchy Sequences and Complete Metric Spaces

Any sequence that **does** converge must be Cauchy:

**Theorem 5** (Thm. 7.2). *Every convergent sequence in a metric space is Cauchy.*

*Proof.* We just did it: Let  $x_n \rightarrow x$ . For every  $\varepsilon > 0 \exists N$  such that  $n > N \Rightarrow d(x_n, x) < \varepsilon/2$ . Then

$$m, n > N \Rightarrow d(x_n, x_m) \leq d(x_n, x) + d(x, x_m) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

□

**Example:** Let  $X = (0, 1]$  and  $d$  be the Euclidean metric. Let  $x_n = \frac{1}{n}$ . Then  $x_n \rightarrow 0$  in  $\mathbf{E}^1$ , so  $\{x_n\}$  is Cauchy in  $\mathbf{E}^1$ . Thus  $\{x_n\}$  is Cauchy in  $(X, d)$ . But  $\{x_n\}$  does not converge in  $(X, d)$ .

The Cauchy property depends only on the sequence and the metric  $d$ , not on the ambient metric space:

$\{x_n\}$  is Cauchy in  $(X, d)$ , but  $\{x_n\}$  does not **converge** in  $(X, d)$  because the point it is trying to converge to (0) is not an element of  $X$ .



# Complete Metric Spaces and Banach Spaces

Where does every Cauchy sequence get what it wants?

**Definition 3.** *A metric space  $(X, d)$  is complete if every Cauchy sequence  $\{x_n\} \subseteq X$  converges to a limit  $x \in X$ .*

**Definition 4.** *A Banach space is a normed vector space that is complete in the metric generated by its norm.*

# Complete Metric Spaces and Banach Spaces

**Example:** Consider the earlier example of  $X = (0, 1]$  with  $d$  the usual Euclidean metric. The sequence  $\{x_n\}$  with  $x_n = \frac{1}{n}$  is Cauchy but does not converge, so  $((0, 1], d)$  is not complete.

**Example:**  $\mathbb{Q}$  is not complete in the Euclidean metric. To see this, let

$$x_n = \frac{\lfloor 10^n \sqrt{2} \rfloor}{10^n}$$

where  $\lfloor y \rfloor$  is the greatest integer less than or equal to  $y$ ;  $x_n$  is just equal to the decimal expansion of  $\sqrt{2}$  to  $n$  digits past the decimal point. Clearly,  $x_n$  is rational.  $|x_n - \sqrt{2}| \leq 10^{-n}$ , so  $x_n \rightarrow \sqrt{2}$  in  $\mathbb{E}^1$ , so  $\{x_n\}$  is Cauchy in  $\mathbb{E}^1$ , hence Cauchy in  $\mathbb{Q}$ ; since  $\sqrt{2} \notin \mathbb{Q}$ ,  $\{x_n\}$  is not convergent in  $\mathbb{Q}$ , so  $\mathbb{Q}$  is not complete.

# Complete Metric Spaces and Banach Spaces

**Theorem 6** (Thm. 7.10).  $\mathbf{R}$  is complete with the usual metric (so  $\mathbf{E}^1$  is a Banach space).

*Proof.* Suppose  $\{x_n\}$  is a Cauchy sequence in  $\mathbf{R}$ . Fix  $\varepsilon > 0$ . Find  $N(\varepsilon/2)$  such that

$$n, m > N(\varepsilon/2) \Rightarrow |x_n - x_m| < \frac{\varepsilon}{2}$$

Let

$$\begin{aligned}\alpha_n &= \sup\{x_k : k \geq n\} \\ \beta_n &= \inf\{x_k : k \geq n\}\end{aligned}$$

Fix  $m > N(\varepsilon/2)$ . Then

$$\begin{aligned}k \geq m &\Rightarrow k > N(\varepsilon/2) \Rightarrow x_k < x_m + \frac{\varepsilon}{2} \\ &\Rightarrow \alpha_m = \sup\{x_k : k \geq m\} \leq x_m + \frac{\varepsilon}{2}\end{aligned}$$

Since  $\alpha_m < \infty$ ,

$$\limsup x_n = \lim_{n \rightarrow \infty} \alpha_n \leq \alpha_m \leq x_m + \frac{\varepsilon}{2}$$

since the sequence  $\{\alpha_n\}$  is decreasing. Similarly,

$$\liminf x_n \geq x_m - \frac{\varepsilon}{2}$$

Therefore,

$$0 \leq \limsup_{n \rightarrow \infty} x_n - \liminf_{n \rightarrow \infty} x_n \leq \varepsilon$$

Since  $\varepsilon$  is arbitrary,

$$\limsup_{n \rightarrow \infty} x_n = \liminf_{n \rightarrow \infty} x_n \in \mathbf{R}$$

Thus  $\lim_{n \rightarrow \infty} x_n$  exists and is real, so  $\{x_n\}$  is convergent. □

# Complete Metric Spaces and Banach Spaces

**Theorem 7** (Thm. 7.11).  $\mathbf{E}^n$  is complete for every  $n \in \mathbf{N}$ .

*Proof.* See de la Fuente.



# Complete Metric Spaces and Banach Spaces

**Theorem 8** (Thm. 7.9). *Suppose  $(X, d)$  is a complete metric space and  $Y \subseteq X$ . Then  $(Y, d) = (Y, d|_Y)$  is complete if and only if  $Y$  is a closed subset of  $X$ .*

*Proof.* Suppose  $(Y, d)$  is complete. We need to show that  $Y$  is closed. Consider a sequence  $\{y_n\} \subseteq Y$  such that  $y_n \rightarrow_{(X, d)} x \in X$ . Then  $\{y_n\}$  is Cauchy in  $X$ , hence Cauchy in  $Y$ ; since  $Y$  is complete,  $y_n \rightarrow_{(Y, d)} y$  for some  $y \in Y$ . Therefore,  $y_n \rightarrow_{(X, d)} y$ . By uniqueness of limits,  $y = x$ , so  $x \in Y$ . Thus  $Y$  is closed.

Conversely, suppose  $Y$  is closed. We need to show that  $Y$  is complete. Let  $\{y_n\}$  be a Cauchy sequence in  $Y$ . Then  $\{y_n\}$  is Cauchy in  $X$ , hence convergent, so  $y_n \rightarrow_{(X, d)} x$  for some  $x \in X$ . Since  $Y$  is closed,  $x \in Y$ , so  $y_n \rightarrow_{(Y, d)} x \in Y$ . Thus  $Y$  is complete.  $\square$

# Complete Metric Spaces and Banach Spaces

**Theorem 9** (Thm. 7.12). *Given  $X \subseteq \mathbf{R}^n$ , let  $C(X)$  be the set of bounded continuous functions from  $X$  to  $\mathbf{R}$  with*

$$\|f\|_{\infty} = \sup\{|f(x)| : x \in X\}$$

*Then  $C(X)$  is a Banach space.*

# Contractions

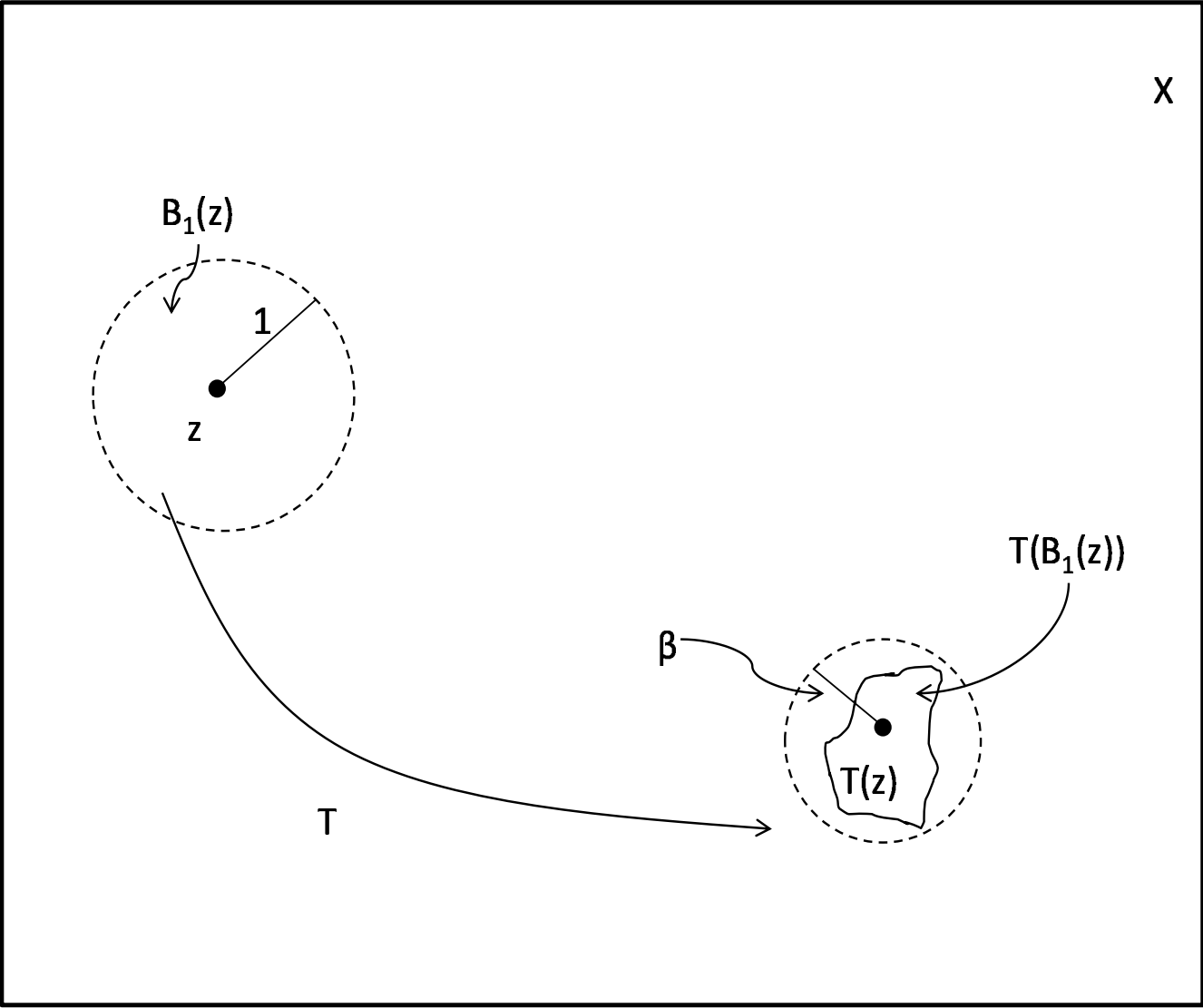
**Definition 5.** Let  $(X, d)$  be a nonempty complete metric space. An operator is a function  $T : X \rightarrow X$ .

An operator  $T$  is a contraction of modulus  $\beta$  if  $0 \leq \beta < 1$  and

$$d(T(x), T(y)) \leq \beta d(x, y) \quad \forall x, y \in X$$

A contraction shrinks distances by a **uniform** factor  $\beta < 1$ .





# Contractions

**Theorem 10.** *Every contraction is uniformly continuous.*

*Proof.* Fix  $\varepsilon > 0$ . Let  $\delta = \frac{\varepsilon}{\beta}$ . Then  $\forall x, y$  such that  $d(x, y) < \delta$ ,

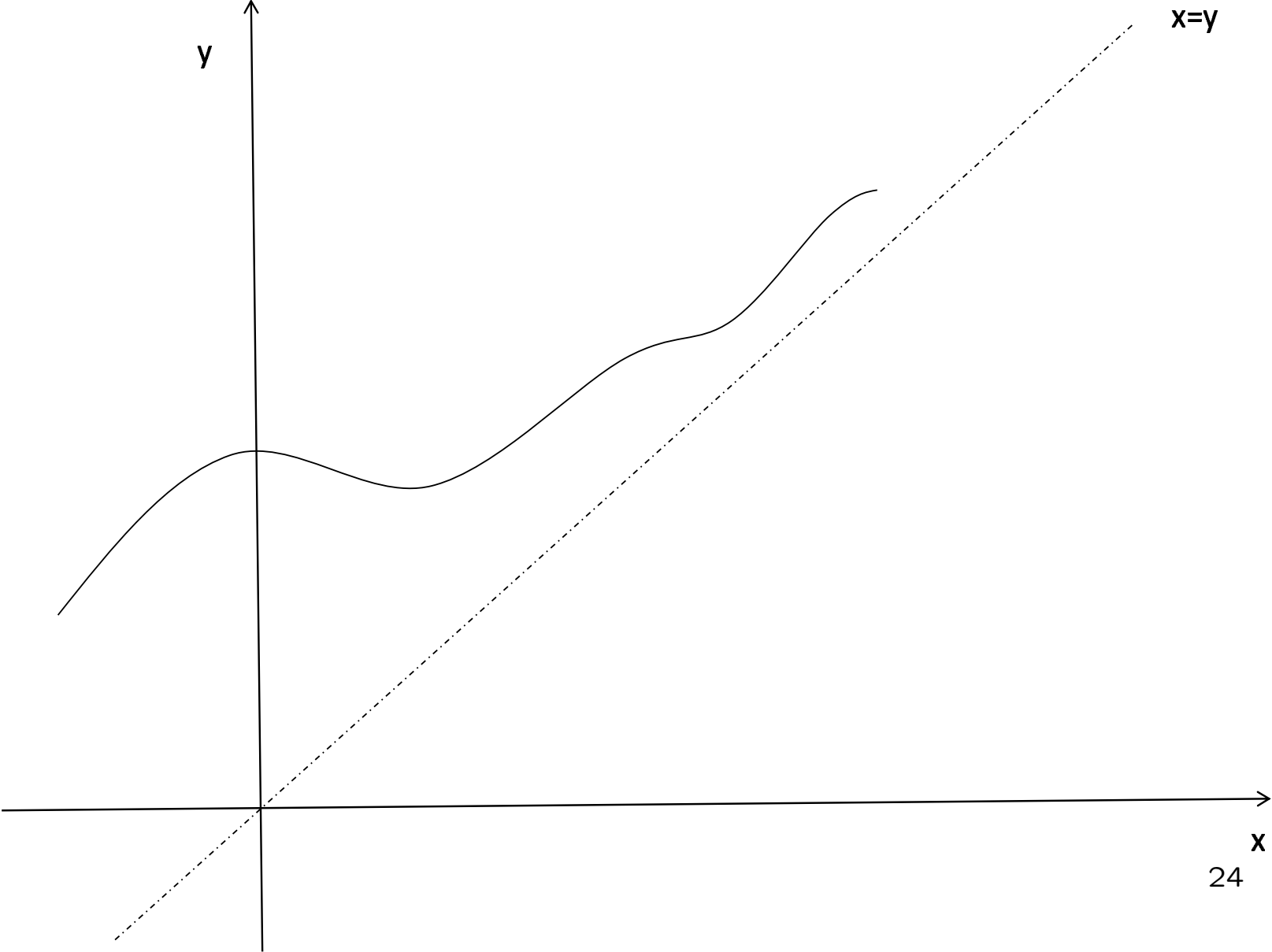
$$d(T(x), T(y)) \leq \beta d(x, y) < \beta \delta = \varepsilon$$

□

Note that a contraction is Lipschitz continuous with Lipschitz constant  $\beta < 1$  (and hence also uniformly continuous).

# Contractions and Fixed Points

**Definition 6.** A fixed point of an operator  $T$  is point  $x^* \in X$  such that  $T(x^*) = x^*$ .



# Contraction Mapping Theorem

**Theorem 11** (Thm. 7.16, Contraction Mapping Theorem). *Let  $(X, d)$  be a nonempty complete metric space and  $T : X \rightarrow X$  a contraction with modulus  $\beta < 1$ . Then*

1.  *$T$  has a unique fixed point  $x^*$ .*

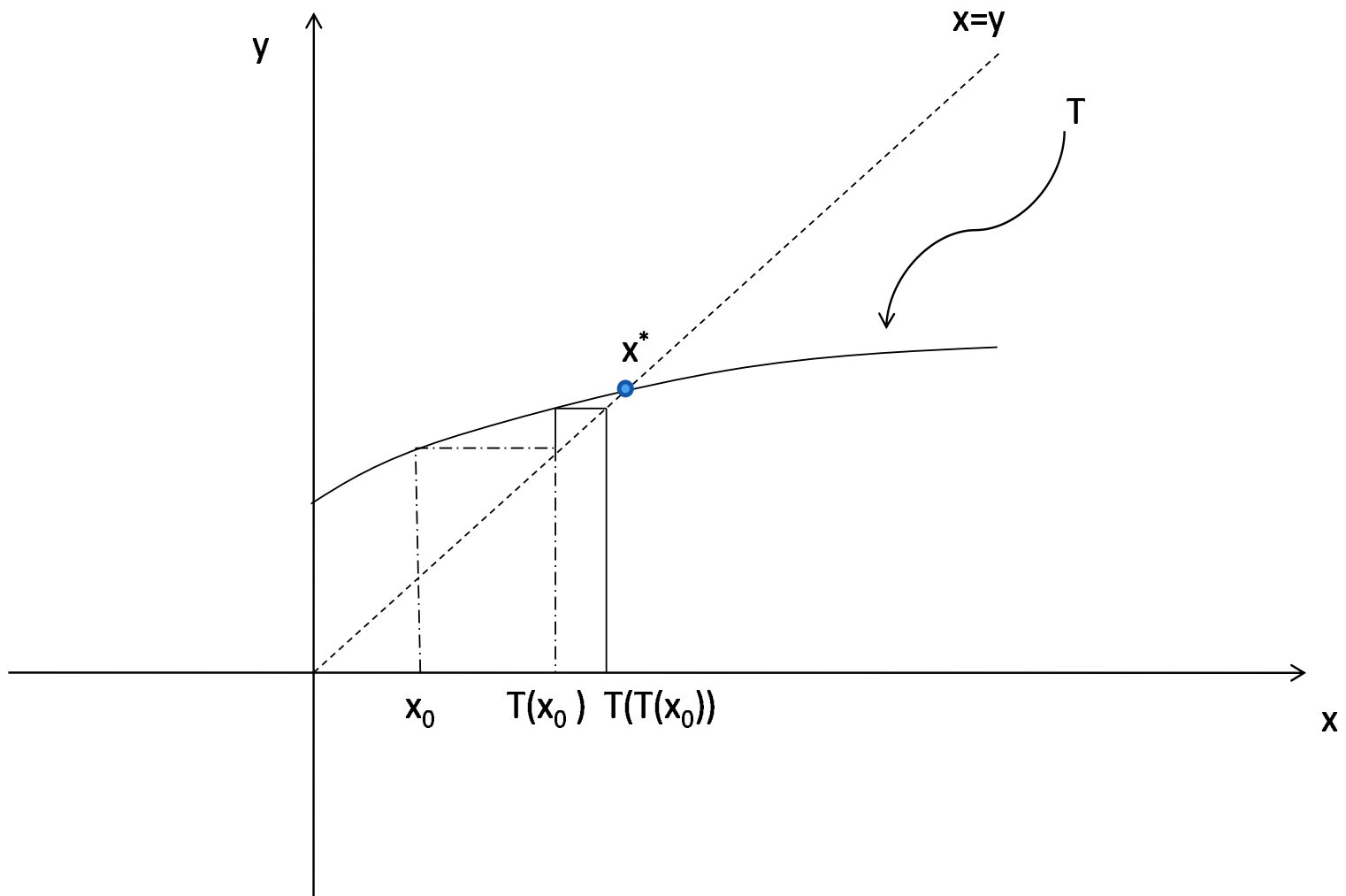
2. *For every  $x_0 \in X$ , the sequence  $\{x_n\}$  where*

*$x_1 = T(x_0), x_2 = T(x_1) = T(T(x_0)), \dots, x_n = T(x_{n-1})$  for each  $n$  converges to  $x^*$ .*

Note that the theorem asserts both the existence and uniqueness of the fixed point, as well as giving an algorithm to find the fixed point of a contraction.

Also note that the algorithm generates a sequence that converges to the fixed point for any initial point  $x_0$ .

Later in the course we will discuss more general fixed point theorems which, in contrast, only guarantee existence, and are not constructive.



*Proof.* Define the sequence  $\{x_n\}$  as above by first fixing  $x_0 \in X$  and then letting  $x_n = T(x_{n-1}) = T^n(x_0)$  for  $n = 1, 2, \dots$ , where  $T^n = T \circ T \circ \dots \circ T$  is the  $n$ -fold iteration of  $T$ . We first show that  $\{x_n\}$  is Cauchy, and hence converges to a limit  $x$ . Then

$$\begin{aligned} d(x_{n+1}, x_n) &= d(T(x_n), T(x_{n-1})) \\ &\leq \beta d(x_n, x_{n-1}) = \beta d(T(x_{n-1}), T(x_{n-2})) \\ &\leq \beta^2 d(x_{n-1}, x_{n-2}) \\ &\vdots \\ &\leq \beta^n d(x_1, x_0) \end{aligned}$$



Then for any  $n > m$ ,

$$\begin{aligned}d(x_n, x_m) &\leq d(x_n, x_{n-1}) + d(x_{n-1}, x_{n-2}) + \cdots + d(x_{m+1}, x_m) \\&\leq (\beta^{n-1} + \beta^{n-2} + \cdots + \beta^m)d(x_1, x_0) \\&= d(x_1, x_0) \sum_{\ell=m}^{n-1} \beta^\ell \\&< d(x_1, x_0) \sum_{\ell=m}^{\infty} \beta^\ell \\&= \frac{\beta^m}{1-\beta}d(x_1, x_0) \quad (\text{sum of a geometric series})\end{aligned}$$

Fix  $\varepsilon > 0$ . Since  $\frac{\beta^m}{1-\beta}d(x_1, x_0) \rightarrow 0$  as  $m \rightarrow \infty$ , choose  $N(\varepsilon)$  such that for any  $m > N(\varepsilon)$ ,  $\frac{\beta^m}{1-\beta}d(x_1, x_0) < \varepsilon$ . Then for  $n, m > N(\varepsilon)$ ,

$$d(x_n, x_m) \leq \frac{\beta^m}{1-\beta}d(x_1, x_0) < \varepsilon$$

Therefore,  $\{x_n\}$  is Cauchy. Since  $(X, d)$  is complete,  $x_n \rightarrow x^*$  for some  $x^* \in X$ .

Next, we show that  $x^*$  is a fixed point of  $T$ .

$$\begin{aligned} T(x^*) &= T\left(\lim_{n \rightarrow \infty} x_n\right) \\ &= \lim_{n \rightarrow \infty} T(x_n) \text{ since } T \text{ is continuous} \\ &= \lim_{n \rightarrow \infty} x_{n+1} \\ &= x^* \end{aligned}$$

so  $x^*$  is a fixed point of  $T$ .

Finally, we show that there is at most one fixed point. Suppose  $x^*$  and  $y^*$  are both fixed points of  $T$ , so  $T(x^*) = x^*$  and  $T(y^*) = y^*$ .

Then

$$\begin{aligned}d(x^*, y^*) &= d(T(x^*), T(y^*)) \\ &\leq \beta d(x^*, y^*) \\ \Rightarrow (1 - \beta)d(x^*, y^*) &\leq 0 \\ \Rightarrow d(x^*, y^*) &\leq 0\end{aligned}$$

So  $d(x^*, y^*) = 0$ , which implies  $x^* = y^*$ .



## Continuous Dependence on Parameters

**Theorem 12. (Thm. 7.18', Continuous Dependence on Parameters)** *Let  $(X, d)$  and  $(\Omega, \rho)$  be two metric spaces and  $T : X \times \Omega \rightarrow X$ . For each  $\omega \in \Omega$  let  $T_\omega : X \rightarrow X$  be defined by*

$$T_\omega(x) = T(x, \omega)$$

*Suppose  $(X, d)$  is complete,  $T$  is continuous in  $\omega$ , that is  $T(x, \cdot) : \Omega \rightarrow X$  is continuous for each  $x \in X$ , and  $\exists \beta < 1$  such that  $T_\omega$  is a contraction of modulus  $\beta \quad \forall \omega \in \Omega$ . Then the fixed point function  $x^* : \Omega \rightarrow X$  defined by*

$$x^*(\omega) = T_\omega(x^*(\omega))$$

*is continuous.*

## Blackwell's Sufficient Conditions

An important result due to Blackwell gives a set of sufficient conditions for an operator to be a contraction that is particularly useful in dynamic programming problems.

Let  $X$  be a set, and let  $B(X)$  be the set of all bounded functions from  $X$  to  $\mathbf{R}$ . Then  $(B(X), \|\cdot\|_\infty)$  is a normed vector space.

Notice that below we use shorthand notation that identifies a constant function with its constant value in  $\mathbf{R}$ , that is, we write interchangeably  $a \in \mathbf{R}$  and  $a : X \rightarrow \mathbf{R}$  to denote the function such that  $a(x) = a \forall x \in X$ .

# Blackwell's Sufficient Conditions

**Theorem 13. (Blackwell's Sufficient Conditions)** Consider  $B(X)$  with the sup norm  $\| \cdot \|_\infty$ . Let  $T : B(X) \rightarrow B(X)$  be an operator satisfying

1. (monotonicity)  $f(x) \leq g(x) \forall x \in X \Rightarrow (Tf)(x) \leq (Tg)(x) \forall x \in X$

2. (discounting)  $\exists \beta \in (0, 1)$  such that for every  $a \geq 0$  and  $x \in X$ ,

$$(T(f + a))(x) \leq (Tf)(x) + \beta a$$

Then  $T$  is a contraction with modulus  $\beta$ .

*Proof.* Fix  $f, g \in B(X)$ . By the definition of the sup norm,

$$f(x) \leq g(x) + \|f - g\|_\infty \quad \forall x \in X$$

Then

$$\begin{aligned} (Tf)(x) &\leq (T(g + \|f - g\|_\infty))(x) \quad \forall x \in X && \text{(monotonicity)} \\ &\leq (Tg)(x) + \beta\|f - g\|_\infty \quad \forall x \in X && \text{(discounting)} \end{aligned}$$

Thus

$$(Tf)(x) - (Tg)(x) \leq \beta\|f - g\|_\infty \quad \forall x \in X$$

Reversing the roles of  $f$  and  $g$  above gives

$$(Tg)(x) - (Tf)(x) \leq \beta\|f - g\|_\infty \quad \forall x \in X$$

Thus

$$\|T(f) - T(g)\|_\infty \leq \beta\|f - g\|_\infty$$

Thus  $T$  is a contraction with modulus  $\beta$

□