Econ 204 2021

Lecture 8

Outline

1. Bases
2. Linear Transformations
3. Isomorphisms

Linear Combinations and Spans

Definition 1. *Let X be a vector space over a field F. A* linear combination *of $x_1, \ldots, x_n \in X$ is a vector of the form*

$$y = \sum_{i=1}^{n} \alpha_i x_i \text{ where } \alpha_1, \ldots, \alpha_n \in F$$

α_i is the coefficient *of x_i in the linear combination.*

If $V \subseteq X$, the span *of V, denoted* $\operatorname{span} V$*, is the set of all linear combinations of elements of V.*

A set $V \subseteq X$ *spans* X if $\operatorname{span} V = X$.

Linear Dependence and Independence

Definition 2. *A set* $V \subseteq X$ *is* linearly dependent *if there exist* $v_1, \ldots, v_n \in V$ *and* $\alpha_1, \ldots, \alpha_n \in F$ *not all zero such that*

$$\sum_{i=1}^{n} \alpha_i v_i = 0$$

A set $V \subseteq X$ *is* linearly independent *if it is not linearly dependent.*

Thus $V \subseteq X$ is linearly independent if and only if

$$\sum_{i=1}^{n} \alpha_i v_i = 0, \quad v_i \in V \ \forall i \Rightarrow \alpha_i = 0 \ \forall i$$

Bases

Definition 3. *A* Hamel basis *(often just called a* basis*) of a vector space X is a linearly independent set of vectors in X that spans X.*

Example: $\{(1,0),(0,1)\}$ is a basis for $\mathbf{R}^2$ (this is the standard basis).

Example, cont: $\{(1,1),(-1,1)\}$ is another basis for $\mathbf{R}^2$:

$$\begin{aligned}
\text{Suppose } (x,y) &= \alpha(1,1)+\beta(-1,1) \text{ for some } \alpha,\beta \in \mathbf{R} \\
x &= \alpha-\beta \\
y &= \alpha+\beta \\
x+y &= 2\alpha \\
\Rightarrow \alpha &= \frac{x+y}{2} \\
y-x &= 2\beta \\
\Rightarrow \beta &= \frac{y-x}{2} \\
(x,y) &= \frac{x+y}{2}(1,1)+\frac{y-x}{2}(-1,1)
\end{aligned}$$

Since (x,y) is an arbitrary element of $\mathbf{R}^2$, $\{(1,1),(-1,1)\}$ spans $\mathbf{R}^2$. If $(x,y)=(0,0)$,

$$\alpha = \frac{0+0}{2} = 0, \quad \beta = \frac{0-0}{2} = 0$$

so the coefficients are all zero, so $\{(1,1),(-1,1)\}$ is linearly independent. Since it is linearly independent and spans $\mathbf{R}^2$, it is a basis.

Example: $\{(1,0,0),(0,1,0)\}$ is not a basis of $\mathbf{R}^3$, because it does not span $\mathbf{R}^3$.

Example: $\{(1,0),(0,1),(1,1)\}$ is not a basis for $\mathbf{R}^2$.

$$1(1,0)+1(0,1)+(-1)(1,1)=(0,0)$$

so the set is not linearly independent.

Bases

Theorem 1 (Thm. 1.2'). *Let V be a Hamel basis for X. Then every vector $x \in X$ has a* unique *representation as a linear combination of a finite number of elements of V (with* all *coefficients nonzero).**

Proof. Let $x \in X$. Since V spans X, we can write

$$x = \sum_{s \in S_1} \alpha_s v_s$$

where S_1 is finite, $\alpha_s \in F$, $\alpha_s \neq 0$, and $v_s \in V$ for each $s \in S_1$. Now, suppose

$$x = \sum_{s \in S_1} \alpha_s v_s = \sum_{s \in S_2} \beta_s v_s$$

*The unique representation of 0 is $0 = \sum_{i \in \emptyset} \alpha_i b_i$.

where S_2 is finite, $\beta_s \in F$, $\beta_s \neq 0$, and $v_s \in V$ for each $s \in S_2$.
Let $S = S_1 \cup S_2$, and define

$$\begin{aligned} \alpha_s = 0 \quad \text{for} \quad s \in S_2 \setminus S_1 \\ \beta_s = 0 \quad \text{for} \quad s \in S_1 \setminus S_2 \end{aligned}$$

Then

$$\begin{aligned} 0 &= x - x \\ &= \sum_{s \in S_1} \alpha_s v_s - \sum_{s \in S_2} \beta_s v_s \\ &= \sum_{s \in S} \alpha_s v_s - \sum_{s \in S} \beta_s v_s \\ &= \sum_{s \in S} (\alpha_s - \beta_s) v_s \end{aligned}$$

Since V is linearly independent, we must have $\alpha_s - \beta_s = 0$, so $\alpha_s = \beta_s$, for all $s \in S$.

$$s \in S_1 \Leftrightarrow \alpha_s \neq 0 \Leftrightarrow \beta_s \neq 0 \Leftrightarrow s \in S_2$$

so $S_1 = S_2$ and $\alpha_s = \beta_s$ for $s \in S_1 = S_2$, so the representation is unique. $\square$

Bases

Theorem 2. *Every vector space has a Hamel basis.*

Proof. The proof uses the Axiom of Choice. Indeed, the theorem is equivalent to the Axiom of Choice. □

Bases

A closely related result, from which you can derive the previous result, shows that any linearly independent set V in a vector space X can be extended to a basis of X.

Theorem 3. *If X is a vector space and $V \subseteq X$ is linearly independent, then there exists a linearly independent set $W \subseteq X$ such that*

$$V \subseteq W \subseteq \operatorname{span} W = X$$

Bases

Theorem 4. *Any two Hamel bases of a vector space X have the same cardinality (are numerically equivalent).*

Proof. The proof depends on the so-called Exchange Lemma, whose idea we sketch. Suppose that $V = \{v_\lambda : \lambda \in \Lambda\}$ and $W = \{w_\gamma : \gamma \in \Gamma\}$ are Hamel bases of X. Remove one vector v_{λ_0} from V, so that it no longer spans (if it did still span, then v_{λ_0} would be a linear combination of other elements of V, and V would not be linearly independent). If $w_\gamma \in \text{span}\,(V \setminus \{v_{\lambda_0}\})$ for every $\gamma \in \Gamma$, then since W spans, $V \setminus \{v_{\lambda_0}\}$ would also span, contradiction. Thus, we can choose $\gamma_0 \in \Gamma$ such that

$$w_{\gamma_0} \notin \text{span}\left(V \setminus \{v_{\lambda_0}\}\right)$$

Because $w_{\gamma_0} \in \operatorname{span} V$, we can write

$$w_{\gamma_0} = \sum_{i=0}^{n} \alpha_i v_{\lambda_i}$$

where α_0, the coefficient of v_{λ_0}, is not zero (if it were, then we would have $w_{\gamma_0} \in \operatorname{span}\left(V \setminus \{v_{\lambda_0}\}\right)$). Since $\alpha_0 \neq 0$, we can solve for v_{λ_0} as a linear combination of w_{γ_0} and $v_{\lambda_1}, \ldots, v_{\lambda_n}$, so

$$\begin{aligned} &\operatorname{span}\left(\left(V \setminus \{v_{\lambda_0}\}\right) \cup \{w_{\gamma_0}\}\right) \\ &\supseteq \operatorname{span} V \\ &= X \end{aligned}$$

so

$$\left(\left(V \setminus \{v_{\lambda_0}\}\right) \cup \{w_{\gamma_0}\}\right)$$

spans X. From the fact that $w_{\gamma_0} \notin \operatorname{span}\left(V \setminus \{v_{\lambda_0}\}\right)$ one can show that

$$\Big(\Big(V \setminus \{v_{\lambda_0}\}\Big) \cup \{w_{\gamma_0}\}\Big)$$

is linearly independent, so it is a basis of X. Repeat this process to exchange every element of V with an element of W (when V is uncountable, this is done by a process called transfinite induction). At the end, we obtain a bijection from V to W, so that V and W are numerically equivalent. $\square$

Dimension

Definition 4. *The* dimension *of a vector space* X, *denoted* $\dim X$, *is the cardinality of any basis of* X.

For $V \subseteq X$, $|V|$ denotes the cardinality of the set V.

Dimension

Example: The set of all $m \times n$ real-valued matrices is a vector space over $\mathbf{R}$. A basis is given by

$$\{E_{ij} : 1 \leq i \leq m, 1 \leq j \leq n\}$$

where

$$\left(E_{ij}\right)_{k\ell} = \begin{cases} 1 & \text{if } k = i \text{ and } \ell = j \\ 0 & \text{otherwise.} \end{cases}$$

The dimension of the vector space of $m \times n$ matrices is mn.

Dimension and Dependence

Theorem 5 (Thm. 1.4)**.** *Suppose* $\dim X = n \in \mathbf{N}$*. If* $V \subseteq X$ *and* $|V| > n$*, then* V *is linearly dependent.*

Dimension and Dependence

Theorem 6 (Thm. 1.5'). *Suppose* $\dim X = n \in \mathbf{N}$, $V \subseteq X$, *and* $|V| = n$.

- *If* V *is linearly independent, then* V *spans* X, *so* V *is a Hamel basis.*

- *If* V *spans* X, *then* V *is linearly independent, so* V *is a Hamel basis.*

Linear Transformations

Definition 5. *Let* X *and* Y *be two vector spaces over the field* F*. We say* $T : X \to Y$ *is a* linear transformation *if*

$$T(\alpha_1 x_1 + \alpha_2 x_2) = \alpha_1 T(x_1) + \alpha_2 T(x_2) \quad \forall x_1, x_2 \in X, \alpha_1, \alpha_2 \in F$$

Let $L(X,Y)$ denote the set of all linear transformations from X to Y.

Linear Transformations

Theorem 7. *$L(X,Y)$ is a vector space over F.*

Proof. First, define linear combinations in $L(X,Y)$ as follows. For $T_1, T_2 \in L(X,Y)$ and $\alpha, \beta \in F$, define $\alpha T_1 + \beta T_2$ by

$$(\alpha T_1 + \beta T_2)(x) = \alpha T_1(x) + \beta T_2(x)$$

We need to show that $\alpha T_1 + \beta T_2 \in L(X,Y)$.

$$\begin{aligned}
&(\alpha T_1 + \beta T_2)(\gamma x_1 + \delta x_2) \\
&\quad = \alpha T_1(\gamma x_1 + \delta x_2) + \beta T_2(\gamma x_1 + \delta x_2) \\
&\quad = \alpha\left(\gamma T_1(x_1) + \delta T_1(x_2)\right) + \beta\left(\gamma T_2(x_1) + \delta T_2(x_2)\right) \\
&\quad = \gamma\left(\alpha T_1(x_1) + \beta T_2(x_1)\right) + \delta\left(\alpha T_1(x_2) + \beta T_2(x_2)\right) \\
&\quad = \gamma\left(\alpha T_1 + \beta T_2\right)(x_1) + \delta\left(\alpha T_1 + \beta T_2\right)(x_2)
\end{aligned}$$

so $\alpha T_1 + \beta T_2 \in L(X, Y)$.

The rest of the proof involves straightforward checking of the vector space axioms. $\square$

Compositions of Linear Transformations

Given $R \in L(X, Y)$ and $S \in L(Y, Z)$, $S \circ R : X \to Z$. We will show that $S \circ R \in L(X, Z)$, that is, the composition of two linear transformations is linear.

$$\begin{aligned}(S \circ R)(\alpha x_1 + \beta x_2) &= S(R(\alpha x_1 + \beta x_2)) \\ &= S(\alpha R(x_1) + \beta R(x_2)) \\ &= \alpha S(R(x_1)) + \beta S(R(x_2)) \\ &= \alpha (S \circ R)(x_1) + \beta (S \circ R)(x_2)\end{aligned}$$

so $S \circ R \in L(X, Z)$.

Kernel and Rank

Definition 6. *Let* $T \in L(X, Y)$*.*

- *The* image *of* T *is* $\text{Im}\, T = T(X)$
- *The* kernel *of* T *is* $\ker T = \{x \in X : T(x) = 0\}$
- *The* rank *of* T *is* $\text{Rank}\, T = \dim(\text{Im}\, T)$

Rank-Nullity Theorem

Theorem 8 (Thms. 2.9, 2.7, 2.6: The Rank-Nullity Theorem)**.** *Let* X *be a finite-dimensional vector space,* $T \in L(X, Y)$*. Then* $\operatorname{Im} T$ *and* $\ker T$ *are vector subspaces of* Y *and* X *respectively, and*

$$\dim X = \dim \ker T + \operatorname{Rank} T$$

Kernel and Rank

Theorem 9 (Thm. 2.13). *$T \in L(X,Y)$ is one-to-one if and only if* $\ker T = \{0\}$.

Proof. Suppose T is one-to-one. Suppose $x \in \ker T$. Then $T(x) = 0$. But since T is linear, $T(0) = T(0 \cdot 0) = 0 \cdot T(0) = 0$. Since T is one-to-one, $x = 0$, so $\ker T = \{0\}$.

Conversely, suppose that $\ker T = \{0\}$. Suppose $T(x_1) = T(x_2)$. Then

$$\begin{aligned} T(x_1 - x_2) &= T(x_1) - T(x_2) \\ &= 0 \end{aligned}$$

which says $x_1 - x_2 \in \ker T$, so $x_1 - x_2 = 0$, so $x_1 = x_2$. Thus, T is one-to-one. □

Invertible Linear Transformations

Definition 7. $T \in L(X, Y)$ *is* invertible *if there exists a function* $S : Y \to X$ *such that*

$$\begin{aligned} S(T(x)) &= x \quad \forall x \in X \\ T(S(y)) &= y \quad \forall y \in Y \end{aligned}$$

Denote S *by* T^{-1}*.*

Note that T is invertible if and only if it is one-to-one and onto. This is just the condition for the existence of an inverse *function.* The linearity of the inverse follows from the linearity of T.

Invertible Linear Transformations

Theorem 10 (Thm. 2.11). *If* $T \in L(X, Y)$ *is invertible, then* $T^{-1} \in L(Y, X)$*, i.e.* T^{-1} *is linear.*

Proof. Suppose $\alpha, \beta \in F$ and $v, w \in Y$. Since T is invertible, there exist unique $v', w' \in X$ such that

$$\begin{array}{rclrcl} T(v') & = & v & T^{-1}(v) & = & v' \\ T(w') & = & w & T^{-1}(w) & = & w' \end{array} .$$

Then

$$\begin{aligned} T^{-1}(\alpha v + \beta w) &= T^{-1}\Big(\alpha T(v') + \beta T(w')\Big) \\ &= T^{-1}\Big(T(\alpha v' + \beta w')\Big) \\ &= \alpha v' + \beta w' \\ &= \alpha T^{-1}(v) + \beta T^{-1}(w) \end{aligned}$$

so $T^{-1} \in L(Y, X)$. $\square$

Linear Transformations and Bases

Theorem 11 (Thm. 3.2)**.** *Let X and Y be two vector spaces over the same field F, and let $V = \{v_\lambda : \lambda \in \Lambda\}$ be a basis for X. Then a linear transformation $T \in L(X,Y)$ is completely determined by its values on V, that is:*

1. *Given any set $\{y_\lambda : \lambda \in \Lambda\} \subseteq Y$, $\exists T \in L(X,Y)$ s.t.*

$$T(v_\lambda) = y_\lambda \quad \forall \lambda \in \Lambda$$

2. *If $S, T \in L(X,Y)$ and $S(v_\lambda) = T(v_\lambda)$ for all $\lambda \in \Lambda$, then $S = T$.*

Proof. 1. If $x \in X$, x has a unique representation of the form

$$x = \sum_{i=1}^{n} \alpha_i v_{\lambda_i} \quad \alpha_i \neq 0 \; i = 1, \ldots, n$$

(Recall that if $x = 0$, then $n = 0$.) Define

$$T(x) = \sum_{i=1}^{n} \alpha_i y_{\lambda_i}$$

Then $T(x) \in Y$. The verification that T is linear is left as an exercise.

2. Suppose $S(v_\lambda) = T(v_\lambda)$ for all $\lambda \in \Lambda$. Given $x \in X$,

$$\begin{aligned} S(x) &= S\left(\sum_{i=1}^{n} \alpha_i v_{\lambda_i}\right) \\ &= \sum_{i=1}^{n} \alpha_i S\left(v_{\lambda_i}\right) \\ &= \sum_{i=1}^{n} \alpha_i T\left(v_{\lambda_i}\right) \\ &= T\left(\sum_{i=1}^{n} \alpha_i v_{\lambda_i}\right) \\ &= T(x) \end{aligned}$$

so $S = T$.

$\square$

Isomorphisms

Definition 8. *Two vector spaces* X *and* Y *over a field* F *are* isomorphic *if there is an invertible* $T \in L(X, Y)$*.*

$T \in L(X, Y)$ *is an* isomorphism *if it is invertible (one-to-one and onto).*

Isomorphic vector spaces are essentially indistinguishable as vector spaces.

Isomorphisms

Theorem 12 (Thm. 3.3). *Two vector spaces X and Y over the same field are isomorphic if and only if* $\dim X = \dim Y$*.*

Proof. Suppose X and Y are isomorphic, and let $T \in L(X, Y)$ be an isomorphism. Let

$$U = \{u_\lambda : \lambda \in \Lambda\}$$

be a basis of X, and let $v_\lambda = T(u_\lambda)$ for each $\lambda \in \Lambda$. Set

$$V = \{v_\lambda : \lambda \in \Lambda\}$$

Since T is one-to-one, U and V have the same cardinality. If

$y \in Y$, then there exists $x \in X$ such that

$$
\begin{aligned}
y &= T(x) \\
&= T\left(\sum_{i=1}^{n} \alpha_{\lambda_i} u_{\lambda_i}\right) \\
&= \sum_{i=1}^{n} \alpha_{\lambda_i} T\left(u_{\lambda_i}\right) \\
&= \sum_{i=1}^{n} \alpha_{\lambda_i} v_{\lambda_i}
\end{aligned}
$$

which shows that V spans Y. To see that V is linearly independent, suppose

$$\begin{aligned} 0 &= \sum_{i=1}^{m} \beta_i v_{\lambda_i} \\ &= \sum_{i=1}^{m} \beta_i T\left(u_{\lambda_i}\right) \\ &= T\left(\sum_{i=1}^{m} \beta_i u_{\lambda_i}\right) \end{aligned}$$

Since T is one-to-one, $\ker T = \{0\}$, so

$$\sum_{i=1}^{m} \beta_i u_{\lambda_i} = 0$$

Since U is a basis, we have $\beta_1 = \cdots = \beta_m = 0$, so V is linearly independent. Thus, V is a basis of Y; since U and V are numerically equivalent, $\dim X = \dim Y$.

Now suppose $\dim X = \dim Y$. Let

$$U = \{u_\lambda : \lambda \in \Lambda\} \text{ and } V = \{v_\lambda : \lambda \in \Lambda\}$$

be bases of X and Y; note we can use the same index set Λ for both because $\dim X = \dim Y$. By Theorem 3.2, there is a unique

$T \in L(X, Y)$ such that $T(u_\lambda) = v_\lambda$ for all $\lambda \in \Lambda$. If $T(x) = 0$, then

$$
\begin{aligned}
0 &= T(x) \\
&= T\left(\sum_{i=1}^{n} \alpha_i u_{\lambda_i}\right) \\
&= \sum_{i=1}^{n} \alpha_i T\left(u_{\lambda_i}\right) \\
&= \sum_{i=1}^{n} \alpha_i v_{\lambda_i} \\
&\Rightarrow \alpha_1 = \cdots = \alpha_n = 0 \text{ since } V \text{ is a basis} \\
&\Rightarrow x = 0 \\
&\Rightarrow \ker T = \{0\} \\
&\Rightarrow T \text{ is one-to-one}
\end{aligned}
$$

If $y \in Y$, write $y = \sum_{i=1}^{m} \beta_i v_{\lambda_i}$. Let

$$x = \sum_{i=1}^{m} \beta_i u_{\lambda_i}$$

Then

$$\begin{aligned} T(x) &= T\left(\sum_{i=1}^{m} \beta_i u_{\lambda_i}\right) \\ &= \sum_{i=1}^{m} \beta_i T(u_{\lambda_i}) \\ &= \sum_{i=1}^{m} \beta_i v_{\lambda_i} \\ &= y \end{aligned}$$

so T is onto, so T is an isomorphism and X, Y are isomorphic. $\square$