Econ 204 – Problem Set 1[1]

Due Friday July 30, 2021

1. Use induction to prove the following:

   (a) For every $r \in \mathbb{N}$ and $x \in [-1, \infty)$, $(1+x)^r \geq 1 + rx$.

   (b) $\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$ for all $n \in \mathbb{N}$.

2. Prove the following statements:

   (a) Let X an infinite set. Prove that there exists $A \subseteq X$ such that A is countable.

   (b) Show that if X is an infinite set, then there is an injection $r : \mathbb{N} \to X$. (Recall from lecture 2 this implies $|\mathbb{N}| \leq |X|$, thus the cardinality of the natural numbers N is less than or equal to the cardinality of any infinite set.)

3. In the following examples, show that the sets A and B are numerically equivalent by finding a specific bijection between the two.

   (a) $A = [0, 1],\ B = [10, 20]$

   (b) $A = [0, 1],\ B = [0, 1)$

   (c) $A = (-1, 1),\ B = \mathbb{R}$

4. **(Dynkins's π-λ system Theorem)**: The goal of this exercise is to prove this theorem, and review/practice some of the set theoretical results.

   Suppose Ω is some arbitrary set (which need not have any topological or algebraic structure). Then, $\mathscr{F} \subset 2^{\Omega}$ as a collection of subsets of Ω is called a **σ-algebra** if it satisfies following properties:

   - $\emptyset \in \mathscr{F}$.
   - For every $A \subset \Omega$ where $A \in \mathscr{F}$, $A^c \in \mathscr{F}$ (A^c refers to the complement of set A, i.e $\Omega \setminus A$).
   - For every *countable* sequence of subsets $\{A_n\}$ where $A_n \in \mathscr{F}$ for all n, $\bigcup_n A_n \in \mathscr{F}$.

   (a) Show that $\Omega \in \mathscr{F}$.

   (b) Prove that $\mathscr{F}$ is closed under countable intersection.

   Two more definitions: first, $\Lambda \subset 2^{\Omega}$ is called a **λ-system** if:

   - $\Omega \in \Lambda$.
   - If $A, B \in \Lambda$ and $A \subset B$, then $B \setminus A \in \Lambda$.
   - If $\{A_n\}$ is an increasing sequence of subsets, i.e $A_1 \subset A_2 \subset \ldots$, with each element being in Λ, then $\bigcup_n A_n \in \Lambda$.

[1] In case of any problems with the solution to the exercises please email brunosmaniotto@berkeley.edu

Second, $\Pi \subset 2^\Omega$ is called a **π-system**, if it is closed under *finite* intersection. Now **assume** Π is a π-system such that $\Pi \subset \Lambda$, where Λ is a λ-system. The Dynkin's theorem which we want to prove states that the smallest σ-algebra containing Π (denoted by $\sigma(\Pi)$) is a subset of Λ. Try to keep on with each step below until the final result drops out:

(c) Let $\lambda(\Pi)$ be the *smallest*[2] λ-system containing Π. Explain why $\lambda(\Pi) \subset \Lambda$. (Hint: note that $\Pi \subset \Lambda$).

(d) Let $B \in \Pi$ and define $\mathcal{A}_B := \{A \subset \Omega : A \cap B \in \lambda(\Pi)\}$. Show that $\mathcal{A}_B$ is itself a λ-system and contains $\lambda(\Pi)$, i.e $\lambda(\Pi) \subset \mathcal{A}_B$.

(e) Now let $A \in \lambda(\Pi)$, and define $\mathcal{B}_A := \{B \subset \Omega : A \cap B \in \lambda(\Pi)\}$. Again, show that $\mathcal{B}_A$ is a λ-system containing $\lambda(\Pi)$.

(f) Use the previous two steps to show $\lambda(\Pi)$ is also a π-system.

(g) Given that we have seen so far that $\lambda(\Pi)$ is both a π-system and a λ-system, deduce that it has to be a σ-algebra.

(h) Conclude that $\sigma(\Pi) \subset \Lambda$.

5. Let $\mathcal{U}$ and $\mathcal{Z}$ be two sets, and $P : \mathcal{U} \times \mathcal{Z} \to \mathbb{R}$ be a bounded function. Define the *upper and lower value functions* as:

$$\begin{aligned} V_+ &= \inf_{u \in \mathcal{U}} \sup_{z \in \mathcal{Z}} P(u, z) \\ V_- &= \sup_{z \in \mathcal{Z}} \inf_{u \in \mathcal{U}} P(u, z) \end{aligned} \tag{1}$$

(a) Show that $V_+ \geq V_-$.

(b) Call any function $\beta : \mathcal{U} \to \mathcal{Z}$ a *strategy* for the maximizing side. Denote the space of all such strategies as $\mathscr{B}$. Prove the following identity, and explain why it is not in contrast with part (a).

$$V_+ = \inf_{u \in \mathcal{U}} \sup_{\beta \in \mathscr{B}} P(u, \beta(u)) = \sup_{\beta \in \mathscr{B}} \inf_{u \in \mathcal{U}} P(u, \beta(u)) \tag{2}$$

6. Let $f : [a, b] \to \mathbb{R}$. The set $P = \{x_0, x_1, \ldots, x_n\}$ is called a *partition* for $[a, b]$, if $a = x_0 < x_1 < \ldots < x_n = b$. Define $V(f; P) := \sum_{j=1}^{n} \left| f(x_j) - f(x_{j-1}) \right|$. The *variation* of f on $[a, b]$ is defined as

$$V(f; [a, b]) := \sup \left\{ V(f; P) : P \text{ is a partition for } [a, b] \right\}. \tag{3}$$

When $V(f; [a, b])$ is finite, we say that f is of *bounded variation* on $[a, b]$.

(a) Show that the class of functions of bounded variation on $[a, b]$ is closed under addition. That is if f and g have bounded variation on $[a, b]$, then $f + g$ also has bounded variation on $[a, b]$.

(b) Show that if f is of bounded variation on $[a, b]$ and $a \leq c \leq b$, then

$$V(f; [a, b]) = V(f; [a, c]) + V(f; [c, b]). \tag{4}$$

[2] By smallest we mean: $\lambda(\Pi) = \bigcap_{\{\Lambda_\alpha \text{ is } \lambda\text{-system in } 2^\Omega : \Pi \subset \Lambda_\alpha\}} \Lambda_\alpha$